A NEW DEFINITION OF THE STEENROD OPERATIONS IN ALGEBRAIC GEOMETRY

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ABSTRACT. The Steenrod operations (mod p) in Chow theory are defined for any prime p for a quasi-projective scheme, without appealing to the results of any domain but Milnor's K-theory. The new definition also gives a direct formula that depends only on the scheme itself. Additionally, basic properties of the operations are proved from the new definition. The idea is based on a construction of M. Rost.

1. INTRODUCTION

In topology, the Steenrod operations were defined in [10] for singular cohomology of topological spaces. For a long time, it was not clear that there was an analog in algebraic geometry of singular cohomology, much less the Steenrod operations. However, these questions were resolved when V. Voevodsky defined these operations in motivic cohomology for use in his proof of the Milnor conjecture. The Chow groups, a special case of motivic cohomology, were left intact by the Steenrod operations; however, at this time there was no elementary definition of the operations on the Chow groups. This was taken care of later, when P. Brosnan defined these operations specifically for Chow groups, avoiding the machinery of motivic cohomology.

However, Brosnan's construction is not without its drawbacks. While the Steenrod operations he defines are defined on the Chow groups, he uses the field of equivariant Chow groups in his definition. Furthermore, his construction involves embedding a scheme into a smooth space and then showing that the construction is independent of the embedding. Of course, neither is a tremendous problem, but ideally, one would like to avoid such concerns in any definition and give a direct formula.

More recently, A. Merkurjev has defined the Steenrod operations (mod 2) in a more elementary way, relying only on Chow theory. The results are more difficult to come by mod p (where p is prime) for a variety of reasons; for example, there may not be p-th roots of unity in the ground field. This paper will define these operations for any prime p over any field of characteristic not equal to p.

1.1. **Outline.** We fix a prime p and a ground field F with char $F \neq p$. In section 2, the action of $G = \mu_p$ on schemes is discussed in an elementary way, in the hope that the reader who has not studied algebraic groups in full generality will understand. Several examples are given, including the action of G on $R^p(X)$ (Example 2.1) for a scheme X. This action will allow us to define the Steenrod operations over a field without p-th roots of unity. The action of G on a deformation variety is discussed in 2.3, which later will allow us to reduce to the case of cones (Corollary 4.10).

In section 3, the concept of a G-torsor is introduced, again in an elementary way. This allows us to introduce the homomorphism $\alpha_T : \mathrm{CH}(T/G)/p \to \mathrm{CH}(T/G)/p$ for any G-torsor $T \to T/G$; this map essentially arises from the fact that G-torsors over a field L are classified by $L^\times/L^{\times p}$. In section 4, the Rost operation is introduced; it is the building block of the Steenrod operations. The construction is essentially as follows: if W is a variety over F with a G-action, then $W \setminus W^G \to (W \setminus W^G)/G$ is a G-torsor; we set $\overline{U}_W = (W \setminus W^G)/G$. The Rost operation ρ_i^W is then defined as the composition $\partial_X^{\overline{U}_W} \circ e(L)^{i-1} \circ \alpha_{\overline{U}_W} : \mathrm{CH}_*(\overline{U}_W)/p \to$

$\mathrm{CH}_{*-i}(W^G)/p$, where L is the canonical line bundle over $\overline{U}_W$. Certain properties of the Rost operation are discussed, including functorialities.

In section 5, a map $P_X : \mathrm{CH}(X) \to \mathrm{CH}((R^p(X)\setminus X)/G)$ is defined that essentially mimics Brosnan's map P_G^n from [1]. At this point we can define the Steenrod operations as follows:

$$S_k^X = (-1)^{n-k}\rho_{(p-1)(n+k)} \circ P_X : \mathrm{CH}_n(X)/p \to \mathrm{CH}_{n-(p-1)k}(X)$$

where $\rho = \rho^{R^p(X)}$ (this is possible as $R^p(X)^G = X$, a fact proved in Example 2.1).

Section 6 gives an explicit calculation of the Steenrod operations, which proves that these operations are the same as Brosnan's. In section 7 some properties are proved; finally, section 8 defines the Steenrod operations of cohomological type and proves further properties.

2. μ_p-ACTIONS ON ALGEBRAIC SCHEMES

In this section we define a μ_p-action on a scheme, give some examples and prove some properties.

2.1. **Definitions and basic properties.** Fix a prime p. Let $X = \mathrm{Spec}\,R$ be an affine scheme over a field F. Suppose that R is a $\mathbb{Z}/p\mathbb{Z}$-graded ring $R = R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1}$. Then the group $G = \mu_p = \mathrm{Spec}\,F[t]/(t^p - 1)$ acts on X: the co-action F-algebra homomorphism

$$\theta : R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1} \to F[t]/(t^p - 1) \otimes (R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1})$$

is defined by the formula

$$\theta(r_0 + r_1 + \cdots + r_{p-1}) = 1 \otimes r_0 + t \otimes r_1 + \cdots + t^{p-1} \otimes r_{p-1}$$

Conversely, a G-action on X is induced by a $\mathbb{Z}/p\mathbb{Z}$-graded ring on R as above. We call such a scheme an *affine G-scheme*. Note that the action is trivial if and only if $R_1 \oplus R_2 \oplus \cdots \oplus R_{p-1} = 0$.

In general, a *G-scheme* is a scheme X together with a G-action on X. If X is a quasi-projective scheme, every pair of points on X belongs to some open affine subscheme. It follows that there is a G-invariant affine covering.

From now on we only consider quasi-projective G-schemes over a field. In the constructions and proofs we may restrict to the class of affine G-schemes.

Definition. Let X be an affine G-scheme; consider the ideal

$$I = \sum_{i+j=p} R_i R_j \subset R_0$$

We call the scheme $\mathrm{Spec}\,R_0/I$ the *fixed-point subscheme* of X, denoted X^G. Notice that

$$R_0/I = R/{<}R_1 \oplus R_2 \oplus \cdots \oplus R_{p-1}{>} = R/(I \oplus R_1 \oplus \cdots \oplus R_{p-1})$$

The natural morphism $\phi : X^G \to X$ satisfies the following universal property: if Y is an affine scheme with trivial G-action, then every G-equivariant morphism $Y \to X$ factors uniquely through ϕ.

Definition. Again let X be an affine G-scheme. Denote the scheme $\mathrm{Spec}\,R_0$ by X/G, called the *quotient* (of X by G). The natural morphism $\pi : X \to X/G$ satisfies the following universal property: if Y is an affine scheme with trivial G-action, then every G-equivariant morphism $X \to Y$ factors uniquely through π.

2.2. **Examples.**

Example 2.1. *The scheme $R^p(X)$.* Consider the algebra $F_p = F[x]/(x^p - 1)$. For any scheme X over F, we write $R^p(X)$ for the scheme $R_{F_p/F}(X_{F_p})$, representing the functor $S \mapsto X(S \otimes_F F_p)$. Note that if $\mathrm{char}(F) \neq p$ and F contains all p-th roots of unity, then $R^p(X)$ is a product of p copies of X.

We have a canonical G-action on $R^p(X)$ defined as follows: to define $G \times R^p(X) \to R^p(X)$, we can define a map $G(S) \times R^p(X)(S) \to R^p(X)(S)$, i.e. $\mu_p(S) \times X(S \otimes F_p) \to X(S \otimes F_p)$. For $\xi \in \mu_p(S)$ we can define $\gamma_\xi : S \otimes F_p \to S \otimes F_p$ via $a \otimes x^m \mapsto a\xi^m \otimes x^m$. Hence our map is of the following form:

$$\mu_p(S) \times X(S \otimes F_p) \quad \to \quad X(S \otimes F_p)$$
$$(\xi, f) \quad \mapsto \quad f \circ (\gamma_\xi)^\#$$

Suppose $\mathrm{char}(F) \neq p$ and F contains a p-th root of unity. Then this action is just the standard action of permuting coordinates on X^p, provided we have selected a distinguished p-th root of unity.

By the generalized Yoneda Lemma ([3] Proposition VI-2), the embedding $X(S) \hookrightarrow X(S \otimes F_p)$ gives rise to a morphism $X \to R^p(X)$. If $\mathrm{char}(F) \neq p$ and F contains a p-th root of unity, this morphism corresponds with the diagonal morphism $X \to X^p$. Hence, if $\mathrm{char}(F) \neq p$, by descent theory, we see that X is identified with $(R^p(X))^G$.

Example 2.2. Let $C = \mathrm{Spec}\, S^\bullet$ be a cone over $X = \mathrm{Spec}\, S^0$. For each $i = 0, 1, \ldots, p-1$, let R_i be the coproduct of all S^j with $j \equiv i \pmod p$. The $\mathbb{Z}/p\mathbb{Z}$-grading on $S^\bullet = R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1}$ gives rise to a G-action on C which we call the *canonical G-action.*

By definition, $C^G = \mathrm{Spec}\, R_0/I$. Since $S^i = (S^1)^i$, it follows that $C^G = \mathrm{Spec}\, S^0$, the image of the zero section of C. We also have that $C/G = \mathrm{Spec}\, R_0$. If C is a line bundle over X, i.e. S^1 is an invertible sheaf and $S^i = (S^1)^{\otimes i}$, then $C/G = C^{\otimes p}$.

2.3. **The deformation variety.** Let $X = \mathrm{Spec}\, R$ be an affine scheme with a G-action; let $Y = X^G = \mathrm{Spec}\,(R/J)$ where $J = I \oplus R_1 \oplus \cdots \oplus R_{p-1}$ as before. Let N be the normal cone of the closed embedding $Y \to X$ and let

$$D = \mathrm{Spec}\, \widetilde{R} = \mathrm{Spec}\,(\cdots \oplus J^2 t^{-2} \oplus J t^{-1} \oplus R \oplus R t \oplus \cdots)$$

be the deformation variety. We have a canonical $\mathbb{Z}/p\mathbb{Z}$-grading on $\widetilde{R}$ induced by the grading on R, giving t the zero grading. Explicitly, for each $i = 0, 1, \ldots p-1$, we have

$$\widetilde{R}_i = \cdots \oplus (J^2)_i t^{-2} \oplus J_i t^{-1} \oplus R_i \oplus R_i t \oplus \cdots$$

We consider the ideal $\widetilde{I} = \sum_{i+j=p} \widetilde{R}_i \widetilde{R}_j \subset \widetilde{R}_0$. For $n \geq 0$, $\widetilde{I}_n = I$. Hence we turn our attention to $\widetilde{I}_{-k}$ for $k > 0$.

Proposition 2.3. $\widetilde{I}_{-k} = (J^k)_0$ *for each $k > 0$.*

Proof. One implication is clear: $\widetilde{I}_{-k} \subset (J^k)_0$. It is also clear that

$$\sum_{i+j=p} J_i (J^{k-1})_j \subset \widetilde{I}_{-k}$$

so it is enough to show that $(J^k)_0 \subset \sum_{i+j=p} J_i (J^{k-1})_j$. Now

$$(J^k)_0 = (J \cdot J^{k-1})_0 = J_0 \cdot (J^{k-1})_0 + \sum_{i+j=p} J_i (J^{k-1})_j$$

so we are reduced to showing that $J_0 \cdot (J^{k-1})_0 \subset \sum_{i+j=p} J_i (J^{k-1})_j$. We proceed by induction. For $k = 1$, the left-hand side becomes $J_0 \cdot (J^0)_0 = IR_0 = I$ and the right-hand side becomes

$$\sum_{i+j=p} J_i (J^0)_j = \sum_{i+j=p} R_i R_j = I$$

so the base step is complete.

Now assume that $J_0 \cdot (J^{k-1})_0 \subset \sum_{i+j=p} J_i (J^{k-1})_j$. Then

$$
\begin{aligned}
J_0 (J^k)_0 &= J_0 (J \cdot J^{k-1})_0 \\
&= J_0 \cdot \left[J_0 \cdot (J^{k-1})_0 + \sum_{i+j=p} J_i (J^{k-1})_j \right] \\
&\subset J_0 \cdot \left[\sum_{i+j=p} J_i (J^{k-1})_j \right] \subset \sum_{i+j=p} J_i (J^k)_j
\end{aligned}
$$

completing the proof. $\square$

Corollary 2.4. *If D is the deformation variety as above, then $D^G = X^G \times \mathbb{A}^1$.*

Proof. By definition, $D^G = \mathrm{Spec}\,(\widetilde{R}_0/\widetilde{I})$. But we have that

$$
\widetilde{R}_0 = \cdots \oplus (J^2)_0 t^{-2} \oplus J_0 t^{-1} \oplus R_0 \oplus R_0 t \oplus \cdots
$$

and, by Proposition 2.3,

$$
\widetilde{I} = \cdots \oplus (J^2)_0 t^{-2} \oplus J_0 t^{-1} \oplus I \oplus It \oplus \cdots
$$

so

$$
\widetilde{R}_0/\widetilde{I} = (R_0/I) \oplus (R_0/I)t \oplus (R_0/I)t^2 \oplus \cdots
$$

giving us that $D^G = \mathrm{Spec}\,(\widetilde{R}_0/\widetilde{I}) = X^G \times \mathbb{A}^1$. $\square$

3. G-TORSORS AND THE HOMOMORPHISM α

Now that we have introduced the concept of a group action, we move on to G-torsors, the "nice" group actions. We define G-torsors, give some examples and prove some basic properties. Then we can define, for any $T \to T/G$ a G-torsor, the homomorphism α_T which is essential to our main construction.

3.1. **Notation and definition. Notation.** Let $K_*(F)$ be the graded Milnor ring of a field. For a scheme X/F, we set

$$
C(X) = \coprod_{x \in X} K_* F(x)
$$

C(X) is a group, graded by the dimension of $x \in X$. We also set

$$
C_{q,r}(X) = \coprod_{x \in X_{(q)}} K_{q+r} F(x).
$$

The q-th homology group of the complex $C_{q,r}(X)$ is denoted by $A_q(X, K_r)$; these are called the *K-homology groups*. Finally, we define $\mathrm{CH}_n(X) = A_n(X, K_{-n})$ and $\mathrm{CH}(X) = \coprod_n \mathrm{CH}_n(X)$, the *Chow groups of X*.

We will also fix a prime p in this paper and simplify notation as follows: for $\mathrm{CH}_n(X)/p$ and $\mathrm{CH}(X)/p$ we will write $\mathrm{Ch}_n(X)$ and $\mathrm{Ch}(X)$ respectively. Furthermore, instead of $A_p(X, K_r/p)$, we will write $A_{q,r}(X)$.

Lemma 3.1. *Let R be a ring and let $f : M \to N$ be a morphism of projective R-modules. Suppose $f^{\otimes k} : M^{\otimes k} \to N^{\otimes k}$ is surjective for some $k \geq 1$. Then f itself is surjective.*

Proof. Let $\mathfrak{m} \subset R$ be a maximal ideal. When we localize at $\mathfrak{m}, M_{\mathfrak{m}}^{\otimes k} \to N_{\mathfrak{m}}^{\otimes k}$ is now a surjective map of a tensor power of free modules. Hence $M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ must be surjective. Now, since this is true for all maximal ideals $\mathfrak{m}$, $M \to N$ must be surjective. $\square$

Proposition 3.2. *Let $X = \operatorname{Spec} R = \operatorname{Spec}(R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1})$ be an affine G-scheme. Let $I = \sum_{i+j=p} R_i R_j \subset R_0$ as before. Then the following are equivalent:*

(1) $X^G = \emptyset$

(2) $I = R_0$

(3) $R_i^p = R_0$ for all $i = 1, 2, \ldots, p-1$.

(4) $R_1^p = R_0$

(5) $R_i \cdot R_{p-i} = R_0$ for each $i = 1, 2, \ldots p - 1$.

(6) The product map $R_k \otimes_{R_0} R_l \to R_{k+l}$ (where the sum is taken mod p) is an isomorphism for all $k, l = 0, 1, 2, \ldots, p - 1$.

(7) The homomorphism $\Phi : R \otimes_{R_0} R \to F[t]/(t^p - 1) \otimes R$ given by

$$a_k \otimes b_l \mapsto t^k \otimes a_k \cdot b_l \quad (a_k \in R_k, b_l \in R_l)$$

is an isomorphism.

Proof. Certain implications are clear: $(1) \Leftrightarrow (2), (5) \Leftrightarrow (6), (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2), (6) \Rightarrow (5)$, and $(7) \Leftrightarrow (6)$. It will hence suffice to show that $(2) \Rightarrow (3)$ and $(3) + (5) \Rightarrow (6)$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$: The result is obvious if $p = 2$, so let p be an odd prime. Choose an index d. We already have $R_d^p \subset R_0$ so it suffices to show that $R_0 \subset R_d^p$.

Let $m = \frac{p-1}{2}$ and $N = p!m$. Then

$$R_0 = R_0^N = \left(\sum_{i+j=p} R_i R_j \right)^N = \sum_{\substack{n_i \geq 0 \, \forall i \\ \sum n_i = N}} c_\alpha \prod_{i+j=p} (R_i R_j)^{n_i}$$

for some constants c_α. Hence it suffices to show

$$\prod_{i+j=p} (R_i R_j)^{n_i} \subset R_d^p$$

for any choice of $(n_1, \ldots, n_m) \in \mathbb{N}^m$ that sum to N.

Choose such a multi-index $(n_1, \ldots, n_m)$. From our choice of N, there exists a k such that $n_k \geq p!$. Hence we have

$$\prod_{i+j=p} (R_i R_j)^{n_i} = \prod_{\substack{i+j=p \\ i \neq k}} (R_i R_j)^{n_i} (R_k R_{p-k})^{n_k} \subset (R_k R_{p-k})^{n_k} \subset (R_k R_{p-k})^{p!}$$

There exist $m_1, m_2 \in \{1, 2, \ldots, p - 1\}$ (specifically, the elements in $(\mathbb{Z}/p\mathbb{Z})^\times$ such that $km_1 = d$ and $(p - k)m_2 = d$) such that $(R_k)^{m_1} \subset R_d$ and $(R_{p-k})^{m_2} \subset R_d$. Thus:

$$(R_k R_{p-k})^{p!} = [(R_k^{m_1})^{\frac{(p-1)!}{m_1}} (R_{p-k}^{m_2})^{\frac{(p-1)!}{m_2}}]^p \subset [(R_d)^{\frac{(p-1)!}{m_1}} (R_d)^{\frac{(p-1)!}{m_2}}]^p \subset R_d^p$$

the desired result.

$(\mathbf{3}) + (\mathbf{5}) \Rightarrow (\mathbf{6})$: Take $\sum x_i \otimes y_i$ in the kernel of the product map, i.e. $\sum x_i y_i = 0$. Choose $a_j \in R_k$ and $b_j \in R_{p-k}$ such that $\sum a_j b_j = 1$. Then $b_j x_i \in R_0$ and so

$$\sum x_i \otimes y_i = \sum a_j b_j x_i \otimes y_i = \sum a_j \otimes b_j x_i y_i = 0$$

hence the product map is injective.

Next, note that by slightly modifying the argument above, we can prove that the product map $R_i^{\otimes p} \to R_0$ is injective for all i, and is thus an isomorphism in view of (3). In particular, each R_i is a locally free R_0-module of rank 1, hence projective.

Consider the following diagram:

$$
\begin{array}{ccc}
(R_k \otimes_{R_0} R_l)^{\otimes p} & \longrightarrow & (R_{k+l})^{\otimes p} \\
\downarrow & & \downarrow \\
R_0 \otimes_{R_0} R_0 & \longrightarrow & R_0
\end{array}
$$

The vertical maps are isomorphisms by our previous discussion, and the bottom map is surjective (actually an isomorphism). Hence the top map is a surjection, so by Lemma 3.1, $R_k \otimes_{R_0} R_l \to R_{k+l}$ is surjective, the desired result. $\qquad\square$

Definition. Let X be a G-scheme. We call $f : X \to X/G$ a *G-torsor* if there exists an open affine covering $X/G = \cup U_i$ such that $f^{-1}(U_i)$ satisfies the properties of the previous Proposition.

3.2. **Examples.**

Example 3.3. If X is any scheme, the scheme $G \times X$ has a natural G-action: the map $G \times (G \times X) \to G \times X$ is defined by $(g_1, g_2, x) \mapsto (g_1 g_2, x)$. The second projection $G \times X \to X$ is a G-torsor, known as the *trivial G-torsor*.

Example 3.4. Let X be a G-scheme and $U = X \setminus X^G$. Since U^G is empty, the morphism $U \to U/G$ is a G-torsor.

Example 3.5. Suppose X is an affine G-scheme, $X \to X/G$ a G-torsor, and R_0 is a local ring. Then R_1 is a free R_0-module of rank 1, i.e. $R_1 = bR_0$ for some $b \in R^\times$. We have that, for each $i = 1, 2, \ldots, p-1$, b^i generates R_i (as $b^{\otimes i}$ generates $R_1^{\otimes i}$). Let $a = b^p \in R_0^\times$. Then R is a free module on $e_0, e_1, \ldots, e_{p-1}$, where $e_i = b^i$, and

$$e_i e_j = \begin{cases} e_{i+j}, & i + j \leq p - 1 \\ a e_{i+j-p}, & i + j \geq p \end{cases}$$

that is, $R \cong R_0[t]/(t^p - a)$. Note that a is unique up to a p-th power.

Example 3.6. Let $X \to X/G$ be a G-torsor, X affine. Let $R_0 \to S_0$ be a ring homomorphism and set $S = R \otimes_{R_0} S_0$. Then $(S_1)^p = S_0$ and so the natural morphism $\operatorname{Spec} S \to \operatorname{Spec} S_0$ is a G-torsor. In particular, for every point $y \in X/G$, X_y is a G-torsor over $\operatorname{Spec} F(y)$.

Example 3.7. Let $X \to X/G$ be a G-torsor, X affine. Then each R_i is an invertible R_0-module, that is, each R_i is locally free of rank 1 over R_0. It follows that, in general, $X \to X/G$ is a flat morphism. Since we have in any case that $X \to X/G$ is surjective, $X \to X/G$ is in fact faithfully flat.

This implies the following useful result: suppose $X \to X/G$ and $Y \to Y/G$ are G-torsors; $f : X \to Y$ a morphism. Let $f_G : X/G \to Y/G$ be the induced map on the quotient spaces. Let $\mathcal{P}$ be one of the following properties: proper, flat, smooth, regular embedding or l.c.i morphism. Then if f has property $\mathcal{P}$, by descent theory, so does f_G.

3.3. **Basic properties.**

Proposition 3.8. *Let $T \to T/G$ and $S \to S/G$ be G-torsors, with $f : S \to T$ a closed embedding. Then:*
    (a) The induced map $g : S/G \to T/G$ is a closed embedding, and
    (b) let N_f and N_g be the normal cones of f and g, respectively. Then $N_g = N_f/G$.

Proof. We may assume that all the schemes are affine. Then $T = \operatorname{Spec} R$, $T/G = \operatorname{Spec} R_0$, $S = \operatorname{Spec} R/K$ for some (graded) ideal $K \subset R$ and $S/G = \operatorname{Spec} R_0/K_0$. Hence (a) follows easily. Note that we did not use the assumption that these were torsors.

Further, we have

$$\begin{aligned} N_f &= \operatorname{Spec}\left(R/K \oplus K/K^2 \oplus K^2/K^3 \oplus \cdots\right) \\ N_g &= \operatorname{Spec}\left(R_0/K_0 \oplus K_0/K_0^2 \oplus K_0^2/K_0^3 \oplus \cdots\right) \\ \text{and } N_f/G &= \operatorname{Spec}\left(R_0/K_0 \oplus (K/K^2)_0 \oplus (K^2/K^3)_0 \oplus \cdots\right) \end{aligned}$$

so for (b) it suffices to show $(K^n)_0 = K_0^n$ for each $n \in \mathbb{N}$. We have that $K_0^n \subset (K^n)_0$, so it remains to show $(K^n)_0 \subset K_0^n$.

Take $a_1, a_2, \ldots, a_n \in K$ such that $a = \prod_{i-1}^{n} a_i \in R_0$; a is a typical element of $(K^n)_0$. Write each a_i as $a_i = a_i^0 + a_i^1 + \cdots + a_i^n$ with $a_i^j \in K_j$. Then

$$a = \prod_{i=1}^{n} a_i = \sum_{p | j_1 + \cdots + j_n} a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n}$$

the other terms having cancelled. Hence it is enough to show that whenever $p | j_1 + j_2 + \cdots + j_n$, $K_{j_1} K_{j_2} \cdots K_{j_n} \subset K_0^n$. It follows from Proposition 3.2 that $R_{p-j_1} R_{p-j_2} \cdots R_{p-j_n} = R_0$, so

$$
\begin{aligned}
K_{j_1} K_{j_2} \cdots K_{j_n} &= R_0 K_{j_1} K_{j_2} \cdots K_{j_n} \\
&= (R_{p-j_1} K_{j_1})(R_{p-j_2} K_{j_2}) \cdots (R_{p-j_n} K_{j_n}) \\
&\subset K_0 K_0 \cdots K_0 = K_0^n
\end{aligned}
$$

$\square$

Proposition 3.9. *Suppose* $\mathrm{char}(F) \neq p$ *and* F *contains a (primitive) p-th root of unity. We denote* $G' = G(F) = \mu_p(F)$, *the abstract group of the roots of unity of* F. *Let* $\pi : X \to X/G$ *be a* G-*torsor. Then:*

(1) π^* *induces an isomorphism* $\pi^* : Z(X/G) \to Z(X)^{G'}$.

(2) $\pi_* \circ \pi^* : Z(X/G) \to Z(X/G)$ *is multiplication by* p.

(3) Choose a distinguished generator ξ *of* G' *and let* σ *be the automorphism of* $Z(X)$ *given by* ξ. *Then* $\pi^* \circ \pi_* = 1 + \sigma + \sigma^2 + \cdots + \sigma^{p-1}$.

Proof. Let $Y = X/G$. For every point $y \in Y$, $X_y \to \mathrm{Spec}\, F(y)$ is a G-torsor by Example 3.6. By Example 3.5, $X_y = \mathrm{Spec}\, F(y)[t]/(t^p - r)$ for some $r \in F(y)^\times$. We consider two cases.

Case 1: $t^p - r$ **is irreducible in** $F(y)[t]$. Then $F(y)[t]/(t^p - r)$ is a field and X_y consists of only one point x. Note that $\pi^*([y]) = [x]$ and $\pi_*([x]) = p[y]$. Furthermore, our automorphism σ acts on fibers, implying that $\sigma^i([x]) = [x]$ for $i = 0, 1, \ldots, p-1$. Hence we get (2) and (3) for this case.

Case 2: $t^p - r$ **is not irreducible in** $F(y)[t]$. Here $t^p - r$ must split completely as F contains all roots of unity, so $F(y) = F \times F \times \cdots \times F$ (p times) and X_y has p points $x_1, x_2, \ldots, x_p$. We see that $\pi^*([y]) = [x_1] + [x_2] + \cdots + [x_p]$ and, for each i, $\pi_*([x_i]) = [y]$. So $\pi_* \circ \pi^*([y]) = p[y]$ and $\pi^* \circ \pi_*([x_i]) = [x_1] + [x_2] + \cdots + [x_p] = (1 + \sigma + \sigma^2 + \cdots + \sigma^{p-1})([x_i])$, as σ acts transitively on the fiber. Hence (2) and (3) are proved in this case.

One final thing to note is that cycles of the form $[x]$ and $[x_1] + [x_2] + \cdots + [x_p]$ (as above) generate the group $Z(X)^{G'}$ of G'-invariant cycles. Hence π^* is surjective. But since it is certainly injective, it is an isomorphism. $\square$

Let X be any scheme with a G-action. We have two morphisms $G \times X \to X$; the action morphism a, which (roughly speaking) sends $(g, x) \mapsto g \cdot x$ and the second projection q, which sends $(g, x) \mapsto x$. If $\pi : X \to X/G$ is the projection, we note that $\pi \circ a = \pi \circ q : G \times X \to X/G$, as either way, the corresponding F-algebra homomorphism $R_0 \to F[t]/(t^p - 1) \otimes R$ sends $r_0 \mapsto 1 \otimes r_0$.

Consider the exact sequence

$$0 \to K \to Z(X) \overset{a^* - q^*}{\to} Z(G \times X)$$

where $K = \mathrm{Ker}(a^* - q^*) \subset Z(X)$. We denote this kernel as $Z(X)^G$. Note that if F contains a p-th root of unity, then $Z(X)^G = Z(X)^{G'}$, in the notation of Proposition 3.9.

Definition. We call a cycle $\delta \in Z(X)$ G-*invariant* if $\delta \in Z(X)^G$.

Corollary 3.10. *Suppose* $\mathrm{char}(F) \neq p$; *let* $\pi : X \to X/G$ *be a* G-*torsor. Then* π^* *induces an isomorphism* $Z(X/G) \to Z(X)^G$.

Proof. Since $\pi \circ a = \pi \circ q : G \times X \to X/G$, the composition

$$Z(X/G) \xrightarrow{\pi^*} Z(X) \overset{a^* - q^*}{\to} Z(G \times X)$$

is 0; hence we have a unique map $Z(X/G) \to Z(X)^G$ induced by π^*. Now Proposition 3.9
(1) along with descent theory tell us that this map is an isomorphism. $\qquad\square$

3.4. **The homomorphism α.** Let $T \to T/G$ be a G-torsor. For each $x \in T/G$ we define

$$a_x \in F(x)^\times / F(x)^{\times p}$$

in the following way: consider the fiber diagram

(1)
$$\begin{array}{ccc}
\operatorname{Spec} L_x & \longrightarrow & T \\
\downarrow & & \downarrow \\
\operatorname{Spec} F(x) & \longrightarrow & T/G
\end{array}$$

By Example 3.5, $L_x \cong F(x)[t]/(t^p - a_x)$ for some uniquely determined $a_x \in F(x)^\times / F(x)^{\times p}$.

Definition. For every pair of points $x, y \in T/G$, we define a homomorphism

$$\alpha_y^x : K_n F(x)/p \to K_{n+1} F(y)/p$$

via

$$\{c_1, \ldots, c_n\} \mapsto \begin{cases} \{a_y, c_1, \ldots, c_n\} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

We hence get an endomorphism $\alpha_T : C(T/G) \to C(T/G)$ such that its (x, y)-component
is α_y^x. It has degree 0 with respect to the grading on $C(T/G)$, so we may talk about
$\alpha_T : C_k(T/G) \to C_k(T/G)$.

Proposition 3.11. *Let $T \to T/G$ be a G-torsor. Then $d \circ \alpha_T = -\alpha_T \circ d$.*

Proof. Consider the diagram

$$\begin{array}{ccc}
C_k(T/G) & \xrightarrow{d} & C_{k-1}(T/G) \\
\downarrow{\scriptstyle\alpha_T} & & \downarrow{\scriptstyle\alpha_T} \\
C_k(T/G) & \xrightarrow{d} & C_{k-1}(T/G)
\end{array}$$

The (x, y)-component in both directions is nonzero only if y is a specialization of x.

Let $Z = \overline{\{x\}}$ and consider the local ring $O_{Z,y}$. It is a local ring, so we may apply our
construction and get a corresponding $a_{Z,y} \in O_{Z,y}^\times$ (unique up to p-th power) and $L_{Z,y}$ as
before. Note further that $F(x) = \operatorname{qf}(O_{Z,y})$ and $F(y)$ is its residue field. We have that $a_{Z,y}$
specializes to $a_x \in F(x)^\times$ and $a_y \in F(y)^\times$, hence the statement follows from [4], Proposition
100.4 (1) and Fact 100.8 (3). $\qquad\square$

This proposition allows us to consider

$$\alpha_T : A_m(T/G, K_k/p) \to A_m(T/G, K_{k+1}/p)$$

or, more succinctly,

$$\alpha_T : A_{m,k}(T/G) \to A_{m,k+1}(T/G)$$

When T is clear from context, we will often use the simpler notation α.

Proposition 3.12. *Let $g : T' \to T$ be a G-equivariant morphism of G-varieties, such that
$T \to T/G$ and $T' \to T'/G$ are G-torsors. Let f be the induced map $T'/G \to T/G$. Then:*
(1) If f is proper, then $\alpha_T \circ f_* = f_* \circ \alpha_{T'}$.
(2) If f is flat, then $\alpha_{T'} \circ f^* = f^* \circ \alpha_T$.

Proof. Take $x \in T'/G$, and let $y = f(x) \in T/G$. We have that $f^*(a_y) = a_x$, so (1) follows from the projection formula for Milnor's K-groups ([4], Fact 100.8 (3)) and (2) follows immediately from the definitions. $\square$

Proposition 3.13. *Let $f : T \to Y = T/G$ be a G-torsor and $\alpha_T : A_{m,k}(Y) \to A_{m,k+1}(Y)$ be the associated homomorphism.*

(1) *If $T \to Y$ is a trivial G-torsor, then $\alpha_T = 0$.*
(2) *Suppose $\pi : X \to Y$ is a proper morphism. Let $S = T \times_Y X$ be the fiber product. Then the following diagram is commutative:*

$$
\begin{array}{ccc}
A_{m,k}(X) & \xrightarrow{\alpha_S} & A_{m,k+1}(X) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
A_{m,k}(Y) & \xrightarrow{\alpha_T} & A_{m,k+1}(Y)
\end{array}
$$

Proof. 1. Since $T \to Y$ is a trivial torsor, $T = G \times Y$. Hence in our fiber diagram (1), $\operatorname{Spec} L_x = G \times \operatorname{Spec} F(x) = \operatorname{Spec}(F(x)[t]/(t^p - 1))$, so $a_x = 1$ for all $x \in X$. It follows from the definition of α_T that $\alpha_T = 0$.

2. Note that $S \to X$ is a G-torsor by Example 3.6. It suffices here to prove the commutativity of the diagram

$$
\begin{array}{ccc}
C_l(X) & \xrightarrow{\alpha_S} & C_l(X) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
C_l(Y) & \xrightarrow{\alpha_T} & C_l(Y)
\end{array}
$$

The (x, y)-component in both directions is nonzero only if $y = \pi(x)$. Now the statement follows directly from the projection formula for Milnor's K-groups. $\square$

4. The Rost operation

The Rost operation (so named because it is a generalization of the construction given by M. Rost in [9]) builds on α_T defined above and is the main component of our construction of the Steenrod operations.

4.1. Preliminaries.
Let F a field of characteristic not p, $G = \mu_p$. Let W be a variety over F together with a G-action. Let $X = W^G$ be the fixed-point subscheme, $U_W = W \setminus X$.

Let $\overline{W} = W/G$ be the quotient, $\overline{U}_W \subset \overline{W}$ the image of U. We have seen that the projection $\pi : U_W \to \overline{U}_W$ is a G-torsor. We may consider X as a subscheme of $\overline{W}$, with open complement $\overline{U}_W$.

Lemma 4.1. *Suppose $g : S \to W$ is a G-equivariant closed embedding of G-schemes. Let $f : S^G \to W^G$ be the induced morphism. Then there exists an induced G-equivariant closed embedding $\bar{f} : \overline{U}_S \to \overline{U}_W$. Moreover, if g is a regular closed embedding, then so is $\bar{f}$.*

Proof. Since $S \cap W^G = S^G$, we have an induced map $S \setminus S^G \to W \setminus W^G$. This in turn induces $\bar{f} : \overline{U}_S \to \overline{U}_W$, which is certainly still a closed embedding. Moreover, if g is regular, then so is $S \setminus S^G \to W \setminus W^G$, and so $\bar{f}$ will be regular as a result of example 3.7. $\square$

Lemma 4.2. *Suppose $f : Z \to X$ is a closed embedding, and $U \subset X$ an open subscheme. We consider Z as a closed subscheme of X. Let $g : Z \cap U \to U$ be the closed embedding and $\pi : N_f \to Z$ be the normal cone of Z in X. Let N_g be the normal cone of the closed embedding g. Then $N_g = \pi^{-1}(Z \cap U)$.*

Proof. We may assume that all schemes are affine and that $U \subset X$ is a principal open subscheme. Hence $X = \operatorname{Spec} R, Z = \operatorname{Spec} R/K, U = \operatorname{Spec} R_a$. We find $Z \cap U = \operatorname{Spec} R_a/K_a$,

$$N_f = \operatorname{Spec} \left(R/K \oplus K/K^2 \oplus \cdots \right)$$

and

$$N_g = \operatorname{Spec} \left(R_a/K_a \oplus K_a/K_a^2 \oplus \cdots \right)$$

and so the result is clear. $\qquad\qquad\square$

Lemma 4.3. *Suppose $S \to W$ is a G-equivariant closed embedding of G-schemes with normal cone E. Then $\overline{U}_E$ is the normal cone of the closed embedding $\overline{U}_S \to \overline{U}_W$.*

Proof. We may assume all schemes are affine; i.e., $W = \operatorname{Spec} R$, $S = \operatorname{Spec} R/K$, $W^G = \operatorname{Spec} R/J$, $W^G = \operatorname{Spec} R/(J+K)$, and

$$E = \operatorname{Spec} \left(R/K \oplus K/K^2 \oplus K^2/K^3 \oplus \cdots \right)$$

Now $E^G = \operatorname{Spec} \left(R/(J+K) \oplus T \right)$ for some graded module T that we don't care about; for the purposes of this proof it is enough to notice that $\pi(E^G) = W^G$. Hence we can apply Lemma 4.2 to conclude that U_E is the normal cone of U_S in U_W. Finally, by Proposition 3.8, $\overline{U}_E$ is the normal cone of $\overline{U}_S$ in $\overline{U}_W$. $\qquad\square$

4.2. **Definition and functoriality.**

Definition. Let L be the line bundle on $\overline{U}_W$ induced from $\pi : U_W \to \overline{U}_W$ via the inclusion $\mu_p \hookrightarrow \mathbb{G}_m$; that is, $L = (U_W \times \mathbb{A}^1)/G$ where G acts on U_W as usual and on $\mathbb{A}^1$ via $G \hookrightarrow \mathbb{G}_m$. We call L the *canonical line bundle* on $\overline{U}_W$.

For any $i \in \mathbb{N}$, we define the **i-th Rost operation** $\rho_i^W : A_{k,j}(\overline{U}_W) \to A_{k-i,j+i}(X)$ as the composition

$$A_{k,j}(\overline{U}_W) \xrightarrow{\alpha} A_{k,j+1}(\overline{U}_W) \xrightarrow{e} A_{k-i+1,j+i}(\overline{U}_W) \xrightarrow{\partial} A_{k-i,j+i}(X)$$

where $\alpha = \alpha_{U_W}, e = e(L)^{i-1}$ and $\partial = \partial_X^{\overline{U}_W}$. We can define the *total Rost operation* ρ^W by taking the coproduct over the ith Rost operations.

We will be mostly interested in one special case: when $j = -k$, ρ_i^W is an operation $\operatorname{Ch}_k(\overline{U}_W) \to \operatorname{Ch}_{k-i}(X)$ and ρ^W is an operation $\operatorname{Ch}(\overline{U}_W) \to \operatorname{Ch}(X)$.

The next proposition will require the following two results.

Lemma 4.4. *Let $f : X \to Y$ and $g : Y \to W$ be morphisms of (quasi-projective) schemes. If $g \circ f$ is proper and f is finite and surjective, then g is proper.*

Proof. Take any quasi-projective scheme T, and let $f' : X \times T \to Y \times T$ and $g' : Y \times T \to W \times T$ be the induced morphisms. Take $Z \subset Y \times T$ a closed subvariety. Then $C = (f')^{-1}(Z) \subset X \times T$ is closed and so $gf(C) \subset W \times T$ is closed, as $g \circ f$ is proper. But $gf(C) = g(Z)$, so g is proper, the desired result. $\qquad\square$

Corollary 4.5. *Let $f : X \to Y$ be a proper G-equivariant morphism of G-schemes. Then the induced morphism $f' : X/G \to Y/G$ is also proper.*

Proof. Consider the commutative square

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} \\ X/G & \xrightarrow{\ f'\ } & Y/G \end{array}$$

Now ψ is finite, hence proper, and so $\psi \circ f = f' \circ \phi$ is proper. The result follows from Lemma 4.4. $\qquad\square$

Proposition 4.6. *Let $f : S \to W$ be a closed embedding of G-schemes, and $\bar{f} : \overline{U}_S \to \overline{U}_W$ the induced morphism. Let $X = S^G$ and $Y = W^G$. Then, for any $k, j \in \mathbb{Z}, i \in \mathbb{N}$, the following diagram commutes:*

$$
\begin{array}{ccc}
A_{k,j}(\overline{U}_S) & \xrightarrow{\rho_i^S} & A_{k-i,j+i}(X) \\
\bar{f}_* \downarrow & & \downarrow f_* \\
A_{k,j}(\overline{U}_W) & \xrightarrow{\rho_i^W} & A_{k-i,j+i}(Y)
\end{array}
$$

Proof. We have already seen that $\bar{f}$ is proper (Lemma 4.1). In light of Proposition 3.12 and [4], Proposition 53.3, it remains to show that $\partial \circ \bar{f}_* = f_* \circ \partial$. We consider the triples involved:

$$
\begin{array}{ccccc}
\overline{U}_S & \longrightarrow & S/G & \longleftarrow & X \\
\bar{f} \downarrow & & f' \downarrow & & \downarrow f \\
\overline{U}_W & \longrightarrow & W/G & \longleftarrow & Y
\end{array}
$$

By Corollary 4.5, we see that f' is proper; hence we can apply [4], Proposition 49.33 and we are done. $\square$

4.3. **Pull-backs and the Rost operation.** In general, the Rost operation does not commute with pull-backs. However, we do have such a result in a special case.

Proposition 4.7. *Suppose we have a commutative diagram*

$$
\begin{array}{ccc}
W & \xrightarrow{f} & Y \\
\uparrow & & \uparrow \\
W^G & \xrightarrow{g} & Y^G
\end{array}
$$

where f and g are G-equivariant regular closed embeddings with normal bundles V and E, respectively. Suppose further that $V^G = E$. Then this gives rise to a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Ch}(\overline{U}_Y) & \xrightarrow{\bar{f}^*} & \mathrm{Ch}(\overline{U}_W) \\
\rho \downarrow & & \downarrow \rho \\
\mathrm{Ch}(Y^G) & \xrightarrow{g^*} & \mathrm{Ch}(W^G)
\end{array}
$$

Proof. Recall that by Lemma 4.3, $\overline{U}_V$ is the normal bundle of $\overline{U}_W$ in $\overline{U}_Y$. Let $\bar{f} : \overline{U}_W \to \overline{U}_Y$ be the induced morphism, $q_1 : \overline{U}_V \to \overline{U}_W$ and $q_2 : E \to W^G$ the projections. Then we want to prove commutativity in the following diagram:

$$
\begin{array}{ccccc}
\mathrm{Ch}(\overline{U}_Y) & \xrightarrow{\sigma_{\bar{f}}} & \mathrm{Ch}(\overline{U}_V) & \xrightarrow{(q_1^*)^{-1}} & \mathrm{Ch}(\overline{U}_W) \\
\rho \downarrow & (a) & \rho \downarrow & (b) & \downarrow \rho \\
\mathrm{Ch}(Y^G) & \xrightarrow{\sigma_g} & \mathrm{Ch}(E) & \xrightarrow{(q_2^*)^{-1}} & \mathrm{Ch}(W^G)
\end{array}
$$

Let $q : \mathbb{G}_m \times \overline{U}_Y \to \overline{U}_Y$ and $r : \mathbb{G}_m \times Y^G \to Y^G$ be the projections. Then (a) becomes (after rotating by ninety degrees):

$$
\begin{array}{cccccccc}
A(\overline{U}_Y) & \xrightarrow{\ \alpha\ } & A(\overline{U}_Y) & \xrightarrow{\ e\ } & A(\overline{U}_Y) & \xrightarrow{\ \partial\ } & A(Y^G) \\
\downarrow{\scriptstyle q^*} & (1) & \downarrow{\scriptstyle q^*} & (2) & \downarrow{\scriptstyle q^*} & (3) & \downarrow{\scriptstyle r^*} \\
A(\mathbb{G}_m \times \overline{U}_Y) & \xrightarrow{\ \alpha\ } & A(\mathbb{G}_m \times \overline{U}_Y) & \xrightarrow{\ e\ } & A(\mathbb{G}_m \times \overline{U}_Y) & \xrightarrow{\ \partial\ } & A(\mathbb{G}_m \times Y^G) \\
\downarrow{\scriptstyle \{t\}} & (4) & \downarrow{\scriptstyle \{t\}} & (5) & \downarrow{\scriptstyle \{t\}} & (6) & \downarrow{\scriptstyle \{t\}} \\
A(\mathbb{G}_m \times \overline{U}_Y) & \xrightarrow{\ \alpha\ } & A(\mathbb{G}_m \times \overline{U}_Y) & \xrightarrow{\ e\ } & A(\mathbb{G}_m \times \overline{U}_Y) & \xrightarrow{\ \partial\ } & A(\mathbb{G}_m \times Y^G) \\
\downarrow{\scriptstyle \partial} & (7) & \downarrow{\scriptstyle \partial} & (8) & \downarrow{\scriptstyle \partial} & (9) & \downarrow{\scriptstyle \partial} \\
A(\overline{U}_V) & \xrightarrow{\ \alpha\ } & A(\overline{U}_V) & \xrightarrow{\ e\ } & A(\overline{U}_V) & \xrightarrow{\ \partial\ } & A(E)
\end{array}
$$

Some of these are easy to handle. (1) commutes by Proposition 3.12. (2) commutes by [4], Proposition 53.3 (2). (4) certainly anticommutes, (5) commutes by [4], Propositions 49.7 and 49.16 (we view e as the composition of a push-forward and the inverse to a pull-back). (6) and (7) anticommute by [4], Proposition 49.33, and (9) anticommutes by [4], Proposition 49.36.

For (3), we notice that this arises from the following triples:

$$
\begin{array}{ccccc}
\mathbb{G}_m \times \overline{U}_Y & \longrightarrow & \mathbb{G}_m \times Y/G & \longleftarrow & \mathbb{G}_m \times Y^G \\
\downarrow & & \downarrow & & \downarrow \\
\overline{U}_Y & \longrightarrow & Y/G & \longleftarrow & Y^G
\end{array}
$$

Certainly, all of the vertical morphisms are flat so we can apply [4], Proposition 49.33 (2) to conclude that (3) anticommutes.

In (8), the situation is the following:

$$
\begin{array}{ccccc}
A(\mathbb{G}_m \times \overline{U}_Y) & \xrightarrow{\ s_*\ } & A(M) & \xrightarrow{\ (\phi^*)^{-1}\ } & A(\mathbb{G}_m \times \overline{U}_Y) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
A(\overline{U}_V) & \xrightarrow{\ r_*\ } & A(N) & \xrightarrow{\ (\psi^*)^{-1}\ } & A(\overline{U}_V)
\end{array}
$$

where

$$ M = (\mathbb{G}_m \times U_Y \times \mathbb{A}^1)/G \qquad N = (U_V \times \mathbb{A}^1)/G $$

$$ \phi \left(\updownarrow \right) s \qquad \text{and} \qquad \psi \left(\updownarrow \right) r $$

$$ \mathbb{G}_m \times \overline{U}_Y \qquad\qquad \overline{U}_V $$

are line bundles, with the given maps projections and zero sections. This gives rise to

$$
\begin{array}{ccccc}
M & \longrightarrow & (U_D \times \mathbb{A}^1)/G & \longleftarrow & N \\
\phi\left(\updownarrow\right)s & & \pi\left(\updownarrow\right)z & & \psi\left(\updownarrow\right)r \\
\mathbb{G}_m \times \overline{U}_Y & \longrightarrow & \overline{U}_D & \longleftarrow & \overline{U}_V
\end{array}
$$

where D is the deformation variety corresponding to the embedding $W \to Y$. Now z is proper and π is flat, so we can apply [4], Proposition 49.33 and we prove the commutativity in (8).

Overall, we had five squares commute and four anticommute, so the diagram is commutative. Hence (a) is done.

For (b), it suffices to show the commutativity of

$$
\begin{array}{ccc}
\mathrm{Ch}(\overline{U}_W) & \xrightarrow{q_1^*} & \mathrm{Ch}(\overline{U}_V) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\
\mathrm{Ch}(W^G) & \xrightarrow{q_2^*} & \mathrm{Ch}(E)
\end{array}
$$

By Proposition 3.12 and [4], Proposition 53.3 (2), it remains to show the commutativity of

$$
\begin{array}{ccc}
A(\overline{U}_W) & \xrightarrow{q_1^*} & A(\overline{U}_V) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
A(W^G) & \xrightarrow{q_2^*} & A(E)
\end{array}
$$

We expand to see the triples:

$$
\begin{array}{ccccc}
\overline{U}_V & \longrightarrow & V/G & \longleftarrow & V^G = E \\
\downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle q_2} \\
\overline{U}_W & \longrightarrow & W/G & \longleftarrow & W^G
\end{array}
$$

V/G is a vector bundle over W/G, hence g is flat and we can again apply [4], Proposition 49.33 to finish the proof. $\qquad\square$

Corollary 4.8. *Let $W \to X$ be a vector bundle with $W^G = X$. Let $f : Y \to X$ be a regular closed embedding. Consider the transverse diagram*

$$
\begin{array}{ccc}
W|_Y & \longrightarrow & W \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X
\end{array}
$$

Then $\rho^W \circ f^* = \bar{f}^* \circ \rho^{W|_Y}$.

Proof. The normal bundle of $W|_Y$ in W is the pull-back of the normal bundle of Y in X, so this diagram satisfies the conditions of Proposition 4.7. $\qquad\square$

4.4. **Reduction to the case of cones.**

Definition. Let $X = \mathrm{Spec}\, R = \mathrm{Spec}\,(R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1})$ be an affine G-scheme. We say a function $f : X \to \mathbb{A}^1$ is *G-invariant* if $f \in R_0$. This definition is extended to arbitrary G-schemes in the obvious way.

We only need one property of these objects.

Lemma 4.9. *Let X be a G-scheme, $f : X \to \mathbb{A}^1$ a G-invariant function. Let $Y = V(f) \subset X$. Then:*

(1) *The normal cone of Y in X is $L = Y \times \mathbb{A}^1$.*
(2) *The normal cone of Y^G in X^G is $E = Y^G \times \mathbb{A}^1$.*
(3) *$L^G = E$.*

Proof. Both (1) and (2) are clear. For (3), we may assume that all schemes are affine, so that $X = \mathrm{Spec}\, R = \mathrm{Spec}\, R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1}$, $Y = \mathrm{Spec}\, R/K$ where $K = fR \cong R$, $X^G = \mathrm{Spec}\, R/J$, where $J = I \oplus R_1 \oplus \cdots \oplus R_{p-1}$ as usual, and $Y^G = \mathrm{Spec}\, R/(J+K)$.

Now

$$
L = \mathrm{Spec}\, R/K \oplus K/K^2 \oplus \cdots = \mathrm{Spec}\, R/K[\bar{f}]
$$

where $\bar{f}$ is the image of f under $K \cong R$. Furthermore,

$$E = \operatorname{Spec} R/(J+K) \oplus (J+K)/(J+K)^2 \oplus \cdots = \operatorname{Spec} R/(J+K)[\tilde{f}]$$

where $\tilde{f}$ is the image of f under $J + K \cong R$. Finally, since $f \in R_0$, G acts trivially on f, meaning that $L^G = \operatorname{Spec} R/(J+K)[\tilde{f}]$, the desired result. $\square$

Corollary 4.10. *Let W be a G-scheme, $X = W^G$. Let C be the normal cone of X in W. Then*

$$\rho^W([\overline{U}_W]) = \rho^C([\overline{U}_C]) \in \operatorname{Ch}(X)$$

Proof. Let D be the deformation variety. The closed embeddings $f : C \to D$ and $g : W \to D$ are both regular. Consider the following diagram:

$$
\begin{array}{ccccc}
\operatorname{Ch}(\overline{U}_C) & \xleftarrow{\bar{f}^*} & \operatorname{Ch}(\overline{U}_D) & \xrightarrow{\bar{g}^*} & \operatorname{Ch}(\overline{U}_W) \\
\downarrow{\rho^C} & & \downarrow{\rho^D} & & \downarrow{\rho^W} \\
\operatorname{Ch}(X) & \longleftarrow & \operatorname{Ch}(X \times \mathbb{A}^1) & \longrightarrow & \operatorname{Ch}(X)
\end{array}
$$

Note that $C \subset D$ and $W \subset D$ are both defined by G-invariant functions. Therefore, by Proposition 4.7 and Lemma 4.9, both small squares commute. Consider $[\overline{U}_D] \in \operatorname{Ch}(\overline{U}_D)$. If we follow it left and down, it is mapped to $\rho^C([\overline{U}_C])$. If we follow it right and down, it is mapped to $\rho^W([\overline{U}_W])$. But these two must be equal because the bottom row is equality. $\square$

5. The map P_X and the Steenrod operations

In this section we finally describe the construction of the Steenrod operations. The last component we need is the map P_X, which can be viewed as a generalization of the map P_G^n in [1].

Let us set up some notation. Take a scheme X with trivial G-action; consider $R^p(X)$ and the associated G-action as before. We have seen that $X = (R^p(X))^G$; we set $W = R^p(X)$ and use this example from now on. We simplify notation in what follows: instead of $U_{R^p(X)}$ and $\overline{U}_{R^p(X)}$, whenever convenient, we use the notation U and $\overline{U}$. We also write ρ and ρ_i to stand for $\rho^{R^p(X)}$ and $\rho_i^{R^p(X)}$. If X is not clear from context, we will abuse notation and denote these by $U_X, \overline{U}_X, \rho^X$ and ρ_i^X.

5.1. Twists of a G-action.

Definition. Let $W = \operatorname{Spec} R$ be an affine scheme with a G-action, so that $R = R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1}$. Suppose $R_i \cong R_j$ for all $i, j = 1, 2, \ldots, p-1$. Then for each $k = 1, 2, \ldots, p-1$ we have an isomorphism of R:

$$h_k : (r_0, r_1, \ldots, r_{p-1}) \mapsto (r_0, r_{k'}, \ldots, r_{k'(p-1)})$$

where k' is the inverse of $k \pmod{p}$. This induces an isomorphism $\eta_k : W \to W$, known as a *k-twist of W*.

If in addition W has a canonical G-action, and we act on W by G and then apply a k-twist, we say that we have *twisted the action k times* and that G acts on W with *weight k*. Schemes on which we can apply a k-twist include X^p and $R^p(X)$ for any scheme X, along with cones and vector bundles. The last two have a canonical G-action, so we can talk about twisting the action and the weight of the action. Note that this corresponds with the standard definition of twisting a G-action, which is composing the action map $G \times W \to W$ with $G \times W \to G \times W$ given by $(g, x) \mapsto (g^k, x)$.

Lemma 5.1. *Let X be a scheme, and let $L = (U_X \times \mathbb{A}^1)/G$ be the canonical line bundle over $\overline{U}_X$. Then for any $i, j \in \mathbb{Z}$ and $k = 1, 2, \ldots, p-1$,*

(1) $k \cdot (\eta_k)_* \circ \alpha_{\overline{U}} = \alpha_{\overline{U}} \circ (\eta_k)_* : A_{i,j}(\overline{U}) \to A_{i,j+1}(\overline{U})$

(2) $k \cdot (\eta_k)_* \circ e(L) = e(L) \circ (\eta_k)_* : A_{i,j}(\overline{U}) \to A_{i-1,j+1}(\overline{U})$

(3) $\partial_X^{\overline{U}_X} = \partial_X^{\overline{U}_X} \circ (\eta_k)_* : A_{i,j}(\overline{U}) \to A_{i-1,j}(X)$

Proof. (1) Consider the diagram

$$
\begin{array}{ccc}
C_m(\overline{U}) & \xrightarrow{\alpha_{\overline{U}}} & C_m(\overline{U}) \\
\downarrow{\scriptstyle(\eta_k)_*} & & \downarrow{\scriptstyle(\eta_k)_*} \\
C_m(\overline{U}) & \xrightarrow{\alpha_{\overline{U}}} & C_m(\overline{U})
\end{array}
$$

The (x, y)-component in either direction is nonzero only if $y = \eta_k(x)$. We can write $F(x) = R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1}$ and $F(y) = S_0 \oplus S_1 \oplus \cdots \oplus S_{p-1}$ where $S_i = R_{ik}$ for each i. Now (in the notation of Example 3.5) since $R_i = b_x^i R_0$, we have that $S_i = b_x^{ik} R_0$, and so $a_y = (b_x^k)^p = a_x^k$. The result now follows.

(2) $(\eta_k)^*(L) = L^{\otimes k}$, hence $(\eta_k)_* \circ e(L^{\otimes k}) = e(L) \circ (\eta_k)_*$ (by [4], Proposition 53.3), giving the result.

(3) follows from the commutativity of

$$
\begin{array}{ccccc}
\overline{U}_X & \longrightarrow & R^p(X)/G & \longleftarrow & X \\
\downarrow{\scriptstyle(\eta_k)_*} & & \downarrow{\scriptstyle(\eta_k)_*} & & \| \\
\overline{U}_X & \longrightarrow & R^p(X)/G & \longleftarrow & X
\end{array}
$$

and [4], Proposition 49.33 (1). $\qquad\square$

Proposition 5.2. $\rho_i^X : \overline{U}_X \to X$ *is trivial unless* $p - 1 \,|\, i$.

Proof. Consider the diagram (subscripts omitted for brevity):

$$
\begin{array}{ccccccc}
\mathrm{Ch}(\overline{U}_X) & \xrightarrow{\alpha_{\overline{U}_X}} & A(\overline{U}_X) & \xrightarrow{e(L)^{i-1}} & A(\overline{U}_X) & \xrightarrow{\partial_X^{\overline{U}_X}} & \mathrm{Ch}(X) \\
\downarrow{\scriptstyle(\eta_k)_*} & & \downarrow{\scriptstyle(\eta_k)_*} & & \downarrow{\scriptstyle(\eta_k)_*} & & \| \\
\mathrm{Ch}(\overline{U}_X) & \xrightarrow{\alpha_{\overline{U}_X}} & A(\overline{U}_X) & \xrightarrow{e(L)^{i-1}} & A(\overline{U}_X) & \xrightarrow{\partial_X^{\overline{U}_X}} & \mathrm{Ch}(X)
\end{array}
$$

Both the top and bottom compositions equal ρ_i, and hence by Lemma 5.1, $k^i \cdot \rho_i = \rho_i$. Since this is true for any $k \in (\mathbb{Z}/p\mathbb{Z})^\times$, it must be that $\rho_i = 0$ unless $p - 1 \,|\, i$. $\qquad\square$

5.2. The map P_X. For any scheme X, we set $Z(X; \mathbb{P}^1)$ to be the subgroup of $Z(X \times \mathbb{P}^1)$ generated by the classes of closed subvarieties in $X \times \mathbb{P}^1$ that are dominant over $\mathbb{P}^1$. For a rational point $a \in \mathbb{P}^1$ and a variety $V \subset X \times \mathbb{P}^1$ mapping dominantly to $\mathbb{P}^1$, let $V(a)$ denote the fiber above a. We associate to this the cycle $[V(a)]$. This definition is extended to $Z(X; \mathbb{P}^1)$ by linearity.

Proposition 5.3. *Let γ_0 and γ_∞ be two cycles on a scheme X. Then the classes of γ_0 and γ_∞ in $\mathrm{CH}(X)$ are equal if and only if there exists a cycle $\delta \in Z(X; \mathbb{P}^1)$ such that $\gamma_0 = \delta(0)$ and $\gamma_\infty = \delta(\infty)$*

Proof. See [1], Proposition 4.2 $\qquad\square$

Take a closed subvariety $Z \subset X$. Then $R^p(Z) \subset R^p(X)$ is a closed subvariety such that $[R^p(Z)] \subset Z(R^p(X))^G$. We may extend this by linearity to any cycle $\gamma \in Z(X)$ to get a G-invariant $R^p(\gamma) \in Z(R^p(X))$.

By Corollary 3.10 we get a cycle $\bar{\gamma} \in Z(\overline{U})$ such that $\pi^*(\bar{\gamma}) = R^p(\gamma)|_U$. We hence have a map $Z(X) \to Z(\overline{U})$, taking $\gamma \mapsto \bar{\gamma}$.

Lemma 5.4. *If γ_0 and γ_∞ are rationally equivalent cycles in $Z(X)$, then $\bar{\gamma}_0$ and $\bar{\gamma}_\infty$ are rationally equivalent cycles in $Z(\overline{U})$.*

Proof. Let $Z \subset X \times \mathbb{P}^1$ be a closed subvariety dominant over $\mathbb{P}^1$. Consider the following fiber diagram:

$$
\begin{array}{ccc}
\widetilde{Z} & \subset & R^p(X) \times \mathbb{P}^1 \\
\cap & & \cap \\
R^p(Z) & \subset & R^p(X \times \mathbb{P}^1) \;=\; R^p(X) \times R^p(\mathbb{P}^1)
\end{array}
$$

where $\widetilde{Z} = (R^p(X) \times \mathbb{P}^1) \cap R^p(Z) \subset R^p(X) \times \mathbb{P}^1$ is the fiber product. Notice that over an algebraic closure, $\widetilde{Z} = Z \times_{\mathbb{P}^1} Z \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} Z$. By descent theory, we see that $\widetilde{Z} \to \mathbb{P}^1$ is flat. Therefore, every irreducible component of $\widetilde{Z}$ is dominant over $\mathbb{P}^1$, i.e. the cycle $[\widetilde{Z}]$ belongs to $Z(R^p(X); \mathbb{P}^1)$. By linearity, this construction extends to a map

$$
Z(X; \mathbb{P}^1) \to Z(R^p(X); \mathbb{P}^1), \quad \delta \mapsto \widetilde{\delta}
$$

Let G act on $R^p(X) \times \mathbb{P}^1$ by the standard action on $R^p(X)$ and trivially on $\mathbb{P}^1$. By Proposition 5.3, we have a cycle $\delta \in Z(X; \mathbb{P}^1)$ such that $\delta(0) = \gamma_0$ and $\delta(\infty) = \gamma_\infty$. The associated cycle $\widetilde{\delta} \in Z(R^p(X); \mathbb{P}^1)$ is G-invariant. Since $U \times \mathbb{P}^1$ is a G-torsor over $\overline{U} \times \mathbb{P}^1$, the restriction of the cycle $\widetilde{\delta}$ on $U \times \mathbb{P}^1$ gives rise to a well-defined cycle $\widehat{\delta} \in Z(\overline{U}; \mathbb{P}^1)$, such that

$$
(2) \qquad \widetilde{\delta}|_{U \times \mathbb{P}^1} = q^*(\widehat{\delta})
$$

where $q : U \times \mathbb{P}^1 \to \overline{U} \times \mathbb{P}^1$ is the canonical morphism.

Let $Z \subset \overline{U} \times \mathbb{P}^1$ be a closed subvariety dominant over $\mathbb{P}^1$. We have $\pi^{-1}(Z(a)) = q^{-1}(Z)(a)$ for any rational point a of $\mathbb{P}^1$, where $\pi : U \to \overline{U}$ is the canonical morphism. It follows from [4], Proposition 57.7 that

$$
(3) \qquad \pi^*(\eta(a)) = (q^*\eta)(a)
$$

for every cycle $\eta \in Z(\overline{U}; \mathbb{P}^1)$.

Let $\delta = \sum n_i[W_i]$. Then applying (3) for $\eta = \widehat{\delta}$, we get from (2) that

$$
\begin{aligned}
\pi^*(\widehat{\delta}(a)) &= (q^*\widehat{\delta})(a) = \widetilde{\delta}|_{U \times \mathbb{P}^1}(a) \\
&= \sum n_i [\widetilde{W_i}]|_{U \times \mathbb{P}^1}(a) \\
&= \sum n_i [R^p(W_i(a))]|_U \\
&= R^p(\delta(a))|_U \\
&= \pi^*(\overline{\delta(a)})
\end{aligned}
$$

Since π^* is injective, $\widehat{\delta}(a) = \overline{\delta(a)}$ in $Z(\overline{U})$. In particular, $\widehat{\delta}(0) = \overline{\delta(0)} = \bar{\gamma}_0$ and $\widehat{\delta}(\infty) = \overline{\delta(\infty)} = \bar{\gamma}_\infty$, i.e. the cycles $\bar{\gamma}_0$ and $\bar{\gamma}_\infty$ are rationally equivalent, as desired. $\qquad\square$

By this lemma, we have a well-defined map

$$
P_X : \mathrm{CH}(X) \to \mathrm{CH}(\overline{U}), \quad [\gamma] \mapsto [\bar{\gamma}]
$$

If we like, we can also consider P_X as a map $\mathrm{CH}_n(X) \to \mathrm{CH}_{np}(\overline{U})$ for any n. When X is clear from context, we will simplify notation and denote this map by P. Note that this map is not a homomorphism.

Proposition 5.5. *Let $f : X \to Y$ be a closed embedding. Then $\bar{f}_* \circ P_X = P_Y \circ f_*$. Moreover, if f is a regular embedding, $\bar{f}^* \circ P_Y = P_X \circ \bar{f}^*$.*

Proof. Note that this proposition makes sense in view of Lemma 4.1. The first stated commutativity is easy, as it holds on the level of cycles. For the second, we have by Lemma 4.3 that the normal bundle of $\overline{U}_X$ in $\overline{U}_Y$ is $\overline{U}_N$, where N is the normal bundle of X in Y.

Now the map f^* is defined as the composition $(\pi^*)^{-1} \circ \sigma$ where $\sigma : \mathrm{CH}(\overline{U}_Y) \to \mathrm{CH}(\overline{U}_N)$ is the deformation homomorphism and $\pi : \overline{U}_N \to \overline{U}_Y$ is the projection which induces an isomorphism on the Chow groups.

The map σ is defined on cycles by $\sigma([Z]) = [C_{Z \cap X} Z]$, where $C_{Z \cap X} Z$ is the normal cone of $Z \cap X$ in Z. Hence σ commutes with P on the level of cycles. Furthermore, it is easy to see that P commutes with π^* on the level of cycles, and so we are done. $\qquad \square$

Note. P may not commute with flat pull-backs in general as $\bar{f}$ may not even be defined for a flat morphism f.

5.3. **Definition of the Steenrod operations.**

Definition. We define the *k-th Steenrod operation mod p* to be

$$S_k^X = (-1)^{n+k} \rho_{(p-1)(n+k)} \circ P : \mathrm{Ch}_n(X) \to \ \mathrm{Ch}_{n-(p-1)k}(X)$$

By taking the coproduct over all of these, we can define the *total Steenrod operation* $S^X : \mathrm{Ch}(X) \to \mathrm{Ch}(X)$.

Proposition 5.6. S^X *is a homomorphism.*

Proof. By a restriction-corestriction argument, we may assume that F contains all p-th roots of unity. Let $\pi : U \to \overline{U}$ be the projection. Choose a distinguished p-th root of unity ξ of F and let σ be the automorphism of $Z(U)$ associated to ξ.

For any two cycles $\gamma = \sum n_i[Z_i]$ and $\delta = \sum m_i[Z_i]$ on X we have

$$\pi^*(\overline{\gamma + \delta}) - \pi^*(\overline{\gamma}) - \pi^*(\overline{\delta}) = \sum n_i m_j (1 + \sigma + \cdots + \sigma^{p-1})[Z_i \times Z_j]|_U \in Z(U)$$

Since $\pi^* \circ \pi_* = 1 + \sigma + \cdots + \sigma^{p-1}$ by Proposition 3.9 (3), we have

$$P(\gamma + \delta) - P(\gamma) - P(\delta) \in \mathrm{Im}(\pi_*) \subset \mathrm{CH}(U)$$

so it suffices to show $\rho \circ \pi_* = 0$. Since $\rho = \partial \circ e \circ \alpha$, it is certainly enough to show $\alpha \circ \pi_* = 0$.

Consider the fiber diagram

$$
\begin{array}{ccc}
G \times U & \longrightarrow & U \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
U & \xrightarrow{\ \pi\ } & \overline{U}
\end{array}
$$

where $G \times U$ is the fiber product, as $U \to \overline{U}$ is a G-torsor. By Proposition 3.13 (2), we get the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Ch}(U) & \xrightarrow{\ \alpha_{G \times U}\ } & \mathrm{Ch}(U) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
\mathrm{Ch}(\overline{U}) & \xrightarrow{\ \alpha_U\ } & \mathrm{Ch}(\overline{U})
\end{array}
$$

But by Proposition 3.13 (1), $\alpha_{G \times U} = 0$. Hence $\alpha \circ \pi_* = 0$ and we are done. $\qquad \square$

Proposition 5.7. *Let* $i : X \to Y$ *be a closed embedding. Then* $i_* \circ S^Y = S^X \circ i_*$.

Proof. This is immediate from Propositions 4.6 and 5.5. $\qquad \square$

6. Computation of the Steenrod operations

6.1. **Some generalizations of Chern classes.**

Definition. Let V be a vector bundle of rank n on X and let $1 + c_1 + c_2 + \cdots + c_n = \prod_{i=1}^{n}(1 + x_i)$ so that the x_i are the Chern roots of V.

(1) We set $b(V) = \prod_{i=1}^{n}(1 + x_i^{p-1}) = 1 + b_1 + b_2 + \cdots$ and call it the *b-class*, or *multiplicative Chern class* of V.
(2) Set $\omega(V) = \prod_{i=1}^{n}(1 - x_i^{p-1}) = 1 + \omega_1 + \omega_2 + \cdots$ and call it the *omega-class* of V.

Remarks:

(1) The operation b is multiplicative, i.e. $b(V \oplus W) = b(V)b(W)$ for any vector bundles V and W. Furthermore, $b(L) = 1 + c_1(L)^{p-1}$ for a line bundle L. By the splitting principle, these two properties determine b.
(2) If $p = 2$, $\omega(V) = 1 + c_1 + c_2 + \cdots = c(V) = b(V)$.
(3) Since the series b lies in the subring of R of elements of degree divisible by $p - 1$, b is of the form $b = 1 + b_{p-1} + b_{2(p-1)} + \cdots$ with b_i of degree i. The same can be said of ω. In fact, it is easy to see that $\omega_{k(p-1)} = (-1)^k b_{k(p-1)}$.

Definition. Let $M \to X$ be a G-vector bundle with $M^G = X$. Suppose M has a filtration by G-subbundles

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with each quotient $L_i = M_i/M_{i-1}$ a line bundle. For $i = 1, \ldots n$, let r_i be the weight of L_i as a G-module. Then we set

$$\mu(M) = \prod_{i=1}^{n}(r_i + c_1(L_i))$$

and call it the *mu-class of M*.

Lemma 6.1. *Let $j : V \to E$ be a closed embedding of G-vector bundles over a scheme Y, with $V^G = E^G = Y$. Suppose E/V has a filtration as in the above definition; finally, let $i : \overline{U}_V \to \overline{U}_E$ be the induced closed embedding. Then*

$$\rho^V \circ i^* = \mu(E/V) \circ \rho^E$$

Proof. Assume that $V \subset E$ is a subbundle of corank 1; the general result will follow by induction. Take $\gamma \in \mathrm{Ch}(\overline{U}_E)$. Then by the projection formula and Proposition 4.6,

$$(4) \qquad \rho^V i^*(\gamma) = \rho^E i_* i^*(\gamma) = \rho^E i_*(1 \cdot i^*(\gamma)) = \rho^E(i_*(1) \cdot \gamma) = \rho^E(c_1(L)(\gamma))$$

where L is the line normal bundle of $\overline{U}_V \subset \overline{U}_E$.

Now consider the normal bundle E/V of V in E; let r be its weight as a G-module. We may write it as $E/V \otimes \mathbb{A}^1$ where G acts trivially on E/V and with weight r on $\mathbb{A}^1$. Then we have that $L = E/V \otimes L_E^{\otimes r}$, where $L_E = (U_E \times \mathbb{A}^1)/G$ is the canonical line bundle over $\overline{U}_E$.

Hence, continuing from (4),

$$
\begin{aligned}
\rho^V \circ i^*(\gamma) = \rho^E(c_1(L)(\gamma)) &= \rho^E(rc_1(L_E)(\gamma) + c_1(E/V)(\gamma)) \\
&= r\rho^E(\gamma) + c_1(E/V)\rho^E(\gamma) \\
&= (r + c_1(E/V))\rho^E(\gamma)
\end{aligned}
$$

completing the proof. $\square$

6.2. **The main computation.** At this point we need to introduce the concept of equivari-ant Chow groups. We will not be using them much, so we will simply refer the interested reader to [2] or [1] for further reading. Two well-known facts from this theory will be of most importance:

Fact 1. Set $l = c_1(L) \in A^1 B\mu_p$, where L is the canonical equivariant line bundle over pt_F. Then if X has trivial G-action, $\mathrm{Ch}^G(X) = \mathrm{Ch}(X)[l]$ (cf. [1], Theorem 7.1.)

Fact 2. If $T \to T/G$ is a G-torsor, then $\mathrm{Ch}^G(T) = \mathrm{Ch}(T/G)$.

Definition. Let $E \to X$ be a G-vector bundle on a variety X with trivial G-action such that $E^G = X$. Let $j : E \setminus X \to E$ and $s : E \to X$ be the inclusion and projection, respectively. We set $\Phi = \Phi_{E,X} : \mathrm{Ch}^G(X) \to \mathrm{Ch}(\overline{U}_E)$ to be the composition

$$\Phi : \mathrm{Ch}^G(X) \xrightarrow{s^*} \mathrm{Ch}^G(E) \xrightarrow{j^*} \mathrm{Ch}^G(E \setminus X) = \mathrm{Ch}(\overline{U}_E)$$

Note. Under the association $\mathrm{Ch}^G(X) = \mathrm{Ch}(X)[l]$ from Fact 1,

$$\Phi(\delta \cdot l^i) = c_1(L_E)^i(\psi^*(\delta))$$

where L_E is the canonical line bundle on $\overline{U}_E$ and $\psi : \overline{U}_E \to X$ is the projection.

Notation. For any element $\sigma = a_0 + a_1 l + \cdots + a_n l^n \in \mathrm{Ch}^G(X)$, we write $\epsilon(\sigma) = a_0 + a_1 + \cdots + a_n \in \mathrm{Ch}(X)$.

The following Proposition is the main computation.

Proposition 6.2. *Let $q : V \to X$ be a G-vector bundle over a variety X with $V^G = X$. Take a class $\sigma \in \mathrm{Ch}^G(X)$. Then under the composition*

$$\mathrm{Ch}^G(X) \xrightarrow{\Phi} \mathrm{Ch}(\overline{U}_V) \xrightarrow{\rho} \mathrm{Ch}(X)$$

σ is mapped to $\mu(V)^{-1}(\epsilon(\sigma))$.

Proof. We may assume $\sigma = \eta \cdot l^a$ where $\eta \in \mathrm{Ch}_{k-1}(X)$ for some $k > 0$ and $a \in \mathbb{N} \cup \{0\}$. By the note above, it remains to compute $\rho(c_1(L_V)^a(\psi^*\eta))$ where L_V is the canonical line bundle on $\overline{U}_V$ and $\psi : \overline{U}_V \to X$ is the projection. We proceed by induction on rank V.

Suppose first that rank $V = 1$; then $V = X \times \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ with some weight, say r. We have that $V \setminus X = X \times \mathbb{G}_m$ and $\overline{U}_V = (V \setminus X)/G = (X \times \mathbb{G}_m)/G = X \times \mathbb{G}_m$. Furthermore, L_V comes from $(\mathbb{G}_m \times \mathbb{A}^1)/G \to \mathbb{G}_m/G$, which is a trivial line bundle. Hence L_V itself is trivial and so $\rho(c_1(L_V)^a(\psi^*\eta)) = 0$ unless $a = 0$, in which case it equals $\psi^*\eta$. We now compute each step of the calculation of $\rho(\psi^*\eta)$.

1. $\alpha : \mathrm{Ch}_k(X \times \mathbb{G}_m) \to A_{k,-k+1}(X \times \mathbb{G}_m)$

From the right exact sequence

$$\mathrm{Ch}_k(X) \xrightarrow{0} \mathrm{Ch}_k(X \times \mathbb{A}^1) \to \mathrm{Ch}_k(X \times \mathbb{G}_m) \to 0$$

we find that $\mathrm{Ch}_k(X \times \mathbb{G}_m) \cong \mathrm{Ch}_k(X \times \mathbb{A}^1) \cong \mathrm{Ch}_{k-1}(X)$. Note that under this identification, $\psi^*\eta \in \mathrm{Ch}_k(X \times \mathbb{G}_m)$ corresponds to $\eta \in \mathrm{Ch}_{k-1}(X)$.

Similarly, from the split short exact sequence

$$0 \to A_{k,-k+1}(X \times \mathbb{A}^1) \to A_{k,-k+1}(X \times \mathbb{G}_m) \xrightarrow{\partial} \mathrm{Ch}_{k-1}(X) \to 0$$

we get that $A_{k,-k+1}(X \times \mathbb{G}_m) \cong A_{k,-k+1}(X \times \mathbb{A}^1) \oplus \mathrm{Ch}_{k-1}(X)$. We consider $\alpha : \mathrm{Ch}_{k-1}(X) \to A_{k,-k+1}(X \times \mathbb{A}^1) \oplus \mathrm{Ch}_{k-1}(X)$ and want to find the image of the class η.

Let $R = F[t, t^{-1}]$. The canonical G-action on $\mathbb{G}_m$ corresponds to the canonical grading $R = R_0 \oplus R_1 \oplus \cdots \oplus R_{p-1}$. In the twisted grading $R = R'_0 \oplus R'_1 \oplus \cdots \oplus R'_{p-1}$, $R_i = R'_{ir}$. Hence $R'_1 = R_{r'}$ where r' is the inverse of r mod p. Thus α is represented by $t^{r'}$ where t is the standard coordinate on $\mathbb{A}^1$, and so $\alpha(\eta) = (t^{r'}, \eta)$.

2. $e(L_V)^{j-1} : A_{k,-k+1}(X \times \mathbb{G}_m) \to A_{k-j+1,-k+j}(X \times \mathbb{G}_m)$

We have already noted that L_V is a trivial line bundle. Thus $e(L_V)^{j-1}(\alpha(\eta))$ is nonzero only when $j = 1$, and in that case equals $(t^{r'}, \eta)$.

3. The last step is to compute $\partial(t^{r'}, \eta)$ which equals $r'\eta$. Hence $\rho^V(\psi^*\eta) = r'\eta$. Finally, we see that $\mu(V) = r + c_1(V) = r$, so we are done in this case.

Now suppose the result is true for vector bundles of rank $n-1$. Take a vector bundle V of rank n over X and suppose it has a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \overset{i}{\hookrightarrow} V_n = V$, with each quotient $L_i = V_i/V_{i-1}$ a line bundle on which G acts with weight r_i. Let $\bar{\iota}^* : \overline{U}_{V_{n-1}} \to \overline{U}_{V_n}$ and $\psi' : \overline{U}_{V_{n-1}} \to X$ be the obvious morphisms. Note that

$$(5) \qquad \bar{\iota}^*(c_1(L_V)^a(\psi^*\eta)) = c_1(L_{n-1})^a(\psi'^*\eta)$$

and by Lemma 6.1,

$$(6) \qquad \rho^{V_{n-1}}(\bar{\iota}^*(c_1(L_V)^a(\psi^*\eta))) = \mu(V/V_{n-1})\rho^V(c_1(L_V)^a(\psi^*\eta))$$

The right-hand side of (6) equals $(r_n + c_1(L_n))\rho^V(c_1(L_V)^a(\psi^*\eta))$. By our inductive hypothesis and (5), the left-hand side equals $(\prod_{i=1}^{n-1}(r_i + c_1(L_i)))^{-1}\eta$. Multiplying both sides by $(r_n + c_1(L_n))^{-1}$ now gives the result. $\qquad\square$

Notation. Let $Z \subset X$ be a subvariety, $C \to Z$ be a cone and $s : E \to X$ a vector bundle such that $C \subset E$. Consider the class $[C] \in \mathrm{Ch}^G(E)$; set $C_E^G = (s^*)^{-1}([C])$ to be the corresponding element in $\mathrm{Ch}^G(X)$. Finally, set $C_E \in \mathrm{Ch}(X)$ to equal $\epsilon(C_E^G)$.

Corollary 6.3. *Let X be a variety, $X \to W$ a closed embedding of X into a smooth scheme W with $\dim W = e$. Let $i : Z \to X$ be a closed subvariety and C the normal cone of $Z \to R^p(Z)$. Let E be the restriction to X of the normal bundle of $W \to R^p(W)$. Then*

$$\rho^X([\overline{U}_Z]) = (-1)^e \omega(-T_W|_X)(C_E)$$

Proof. First note that by Corollary 4.10 and Proposition 4.6,

$$\rho^X([\overline{U}_Z]) = i_*\rho^Z([\overline{U}_Z]) = i_*\rho^C([\overline{U}_C]) = \rho^D([\overline{U}_C]) = \rho^E([\overline{U}_C])$$

where D is the normal cone of $X \to R^p(X)$. Second, note that $\Phi(C_E^G) = [\overline{U}_C] \in \mathrm{Ch}(\overline{U}_E)$, so $\rho^E([\overline{U}_C]) = \mu(E)^{-1}\epsilon(C_E^G) = \mu(E)^{-1}(C_E)$ by Proposition 6.2. Therefore it suffices to calculate $\mu(E)$.

Set H to be the cokernel of the embedding $F \to F_p = F[t]/(t^p - 1)$; then $E = T_W|_X \otimes H$. Note that e is the rank of $T_W|_X$ as a vector bundle over X. Suppose $T_W|_X$ has a filtration with quotients M_i ($i = 1, 2, \ldots e$) and H has a filtration with quotients H_j ($j = 1, 2, \ldots p-1$); note that the H_j are trivial line bundles. Hence $T_W|_X \otimes H$ has a filtration with quotients $M_i \otimes H_j$, where G acts with weight j, and so

$$
\begin{aligned}
\mu(T_W|_X \otimes H) &= \prod_{j=1}^{p-1}\prod_{i=1}^{e}(j + c_1(M_i \otimes H_j)) \\
&= \prod_{j=1}^{p-1}\prod_{i=1}^{e}(j + c_1(M_i) + c_1(H_j)) \\
&= \prod_{i=1}^{e}\prod_{j=1}^{p-1}(j + c_1(M_i)) \\
&= \prod_{i=1}^{e}(-1 + c_1(M_i)^{p-1}) \\
&= (-1)^e \prod_{i=1}^{e}(1 - c_1(M_i)^{p-1}) \\
&= (-1)^e \omega(T_W|_X)
\end{aligned}
$$

$\qquad\square$

The following lemma is done in [1] but not explicitly stated in this form.

Lemma 6.4. *Let $Z \subset X$ be a subvariety of dimension n and let $X \to W$ be a closed embedding of X into a smooth scheme W. Let C be the normal cone of $Z \to R^p(Z)$ and E the restriction to X of the normal bundle $W \to R^p(W)$. Then in the expression*

$$C_E^G = \sum_{i=1}^r a_i l^i$$

all terms with i not divisible by $p-1$ are zero. In particular, we may write $C_E = \gamma_0 + \gamma_1 + \cdots + \gamma_m$ where each $\gamma_i \in \mathrm{Ch}_{n-i(p-1)}(X)$.

Proof. By a restriction-corestriction argument, we may assume our ground field F contains all roots of unity, so that $R^p(X) = X^p$. Set $e = \dim W$, let $\gamma = [Z] \in \mathrm{Ch}_n(X)$ and let $\Delta : W \to W^p$ be the diagonal embedding. By [1], Proposition 5.2, there exists a well-defined class $\gamma_G^p \in \mathrm{Ch}_{np}^G(X^p)$ which comes from $\gamma^{\times p} \in \mathrm{Ch}_{np}(X^p)$. Next, note that $\Delta_G^!(\gamma_G^p) = C_E^G \in \mathrm{Ch}^G(X)$ by definition ([1], 3.1.1 and [5], 6.2). Then [1], Theorem 8.3 gives the result.

Finally, note that each $a_{i(p-1)} \in \mathrm{Ch}_{pn+(p-1)(i-e)}(X)$, so for each i we set

$$\gamma_i = a_{(e-n-i)(p-1)} \in \mathrm{Ch}_{n-i(p-1)}(X)$$

and we get that $C_E = \sum \gamma_i$. $\square$

Corollary 6.5. *Let X be a variety, $X \to W$ a closed embedding of X into a smooth scheme W with $\dim W = e$. Let $Z \subset X$ be a closed subvariety of dimension n and C the normal cone of $Z \to R^p(Z)$. Let E be the restriction to X of the normal bundle of $W \to R^p(W)$ and write $C_E = \sum \gamma_i$ as in Lemma 6.4. Set $\widetilde{C}_E = \sum (-1)^{e+n+i} \gamma_i$. Then $S^X([Z]) = b(-T_W|_X)(\widetilde{C}_E)$.*

Proof. Set $\tilde{\gamma}_i = (-1)^{e+n+i} \gamma_i$ for each i. Then by Corollary 6.3 and Lemma 6.4,

$$
\begin{aligned}
S_k^X([Z]) &= (-1)^{n+k} \rho_{(p-1)(n+k)}^X P([Z]) \\
&= (-1)^{n+k} \rho_{(p-1)(n+k)}^X (\overline{U}_Z) \\
&= (-1)^{n+k}(-1)^e \sum_{i+j=k} \omega_{i(p-1)}(-T_W|_X)(\gamma_j) \\
&= (-1)^{n+k}(-1)^e \sum_{i+j=k} (-1)^i b_{i(p-1)}(-T_W|_X)(-1)^{e+n+j}(\tilde{\gamma}_j) \\
&= \sum_{i+j=k} b_{i(p-1)}(-T_W|_X)(\tilde{\gamma}_j) \\
&= (b(-T_W|_X)(\widetilde{C}_E))_k
\end{aligned}
$$

and hence $S^X([Z]) = b(-T_W|_X)(\widetilde{C}_E)$, as desired. $\square$

6.3. An explicit presentation of the class C_E^G.

Proposition 6.6. *Let $q : E \to X$ be a vector bundle of rank n. Let $i : Z \to X$ be a closed subvariety and $C \to Z$ a cone, such that*

$$
\begin{array}{ccc}
C & \longrightarrow & E \\
\downarrow & & \downarrow q \\
Z & \xrightarrow{\ i\ } & X
\end{array}
$$

commutes. Then

$$C_E^G = \sum_{j+k=n} c_j^G(E) i_* s_k^G(C) \in \mathrm{Ch}^G(X)$$

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}(C \oplus \mathbf{1}) & \xrightarrow{\ j\ } & \mathbb{P}(E \oplus \mathbf{1}) \\
\downarrow{\scriptstyle l} & & \downarrow{\scriptstyle r} \\
Z & \xrightarrow{\ i\ } & X
\end{array}
$$

Let L and L' be the canonical line bundles on $\mathbb{P}(E \oplus \mathbf{1})$ and $\mathbb{P}(C \oplus \mathbf{1})$, so that $L' = i^*L$. By the equivariant projective bundle theorem, $j_*[\mathbb{P}(C \oplus \mathbf{1})] \in \mathrm{Ch}^G(\mathbb{P}(E \oplus \mathbf{1}))$ can be written in the form

$$
j_*[\mathbb{P}(C \oplus \mathbf{1})] = \sum_{i=0}^{n} e^G(L)^i (r^*\zeta_i)
$$

for unique elements $\zeta_i \in \mathrm{Ch}^G(X)$. Let c be the rank of C over Z, and $d = n - c$. Then

$$
\begin{aligned}
i_* s_k^G(C) &= i_* l_* e^G(L')^{k+c}([\mathbb{P}(C \oplus \mathbf{1})]) \\
&= r_* j_*(e^G(L')^{k+c}([\mathbb{P}(C \oplus \mathbf{1})])) \\
&= r_* e^G(L)^{k+c}(j_*[\mathbb{P}(C \oplus \mathbf{1})])) \\
&= r_* \sum_{i=0}^{n} e^G(L)^{k+c+i} r^*(\zeta_i) \\
&= \sum_{i=0}^{n} s_{k+i-d}^G(E) \cdot \zeta_i
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\sum_k i_* s_k^G(C) t^k &= \sum_k \sum_i (s_{k+i-d}^G(E) t^{k+i})(\zeta_i t^{-i}) \\
&= (\sum_j s_{j-d}^G(E) t^j)(\sum_i \zeta_i t^{-i})
\end{aligned}
$$

and so

$$
\sum_{i=0}^{n} \zeta_i t^{n-i} = (\sum_j c_j^G(E) t^j)(\sum_k i_* s_k^G(C) t^k)
$$

i.e.

$$
\zeta_i = \sum_{j+k=n-i} c_j^G(E) i_* s_k^G(C).
$$

Let $\phi : E \to \mathbb{P}(E \oplus \mathbf{1})$ be the open embedding. Since the pull-back ϕ^*L is a trivial line bundle over E, we have, for any $\delta \in \mathrm{Ch}(X)$,

$$
(\phi^* \circ e^G(L)^i)(r^*\delta) = e^G(\phi^*L)^i(\phi^* r^*\delta) = e^G(\phi^*L)^i(q^*\delta) = \begin{cases} q^*\delta & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}
$$

Therefore,

$$
\begin{aligned}
[C] &= \phi^*([j_*[\mathbb{P}(C \oplus \mathbf{1})]) \\
&= \phi^*(\sum_{i=0}^{n} e^G(L)^i r^*(\sum_{j+k=n-i} c_j^G(E) i_* s_k^G(C))) \\
&= q^*(\sum_{j+k=n} c_j^G(E) i_* s_k^G(C)) \in \mathrm{Ch}^G(E)
\end{aligned}
$$

and the result follows by the definition of C_E^G. $\square$

Example 6.7. Suppose X is smooth of dimension d and that $C = E = N$, the normal bundle of $X \to R^p(X)$. Then by Proposition 6.6, $C_E^G = (c^G(N)s^G(N))_d = [X] \in \mathrm{Ch}^G(X)$ and so $C_E = \widetilde{C}_E = [X] \in \mathrm{Ch}(X)$.

7. Properties of the Steenrod operations

7.1. **Basic properties.**

Theorem 7.1. $S_k^X = 0$ *if* $k < 0$ *and* S_0^X *is the identity.*

Proof. We first consider $S_k^X([X])$ where X is a variety of dimension d. By dimension count, $S_k^X([X]) \in \mathrm{Ch}_{d-k}(X)$ is trivial if $k < 0$. To compute $S_0^X([X])$, we may replace X by X_{red}, pass to an algebraic closure, and replace X by a smooth open subscheme. Let N be the normal bundle of X in $R^p(X) = X^p$. Then by Corollary 6.5, Proposition 6.6 and Example 6.7,

$$S_0^X([X]) = b_0(-T_X)(\widetilde{N}_N) = [X]$$

In general, let $i : Z \to X$ be a closed subvariety. Then by Proposition 5.7 and the first part of the proof, $S_k^X([Z]) = i_* S_k^Z([Z])$ is trivial for $k < 0$ and is equal to $[Z] \in \mathrm{Ch}(X)$ if $k = 0$. $\qquad\square$

Proposition 7.2. *(Formula for a smooth cycle) Let Z be a smooth closed subvariety of a scheme X. Then*

$$S^X([Z]) = i_* b(-T_Z)([Z])$$

where $i : Z \to X$ is the closed embedding.

Proof. First, assume X is smooth; let d be the dimension of X. Let N be the normal bundle of $X \to R^p(X)$. Then by Corollary 6.5, Proposition 6.6 and Example 6.7:

$$S^X([X]) = b(-T_X)(\widetilde{N}_N) = b(-T_X)([X])$$

In general, if Z is a smooth closed subvariety of a (not necessarily smooth) scheme X, then by Proposition 5.7 and the first part of the proof, we have

$$S^X([Z]) = i_* S^Z([Z]) = i_* b(-T_Z)([Z])$$

$\qquad\square$

Theorem 7.3. *Let X and Y be two schemes. Then $S^{X \times Y}(\gamma \times \delta) = S^X(\gamma) \times S^Y(\delta)$ for any $\gamma \in \mathrm{Ch}(X)$ and $\delta \in \mathrm{Ch}(Y)$.*

Proof. We may assume that $\gamma = [Z]$ and $\delta = [B]$ where $Z \subset X$ is a closed subvariety of dimension n and $B \subset Y$ is a closed subvariety of dimension m.

Let $f : X \to W$ and $g : B \to M$ be closed embeddings of X, Y into smooth schemes W and M of dimensions r and s, respectively. Let C and D be the normal cones of $X \to R^p(X)$ and $Y \to R^p(Y)$, respectively. Let E be the restriction to X of the normal bundle of $W \to R^p(W)$, and similarly, V the restriction to Y of the normal bundle of $M \to R^p(M)$. Note that $f \times g : X \times Y \to W \times M$ is a closed embedding of $X \times Y$ into a smooth scheme of dimension $r + s$, that $E \times V$ is the restriction to $X \times Y$ of the normal bundle of $W \times M \to R^p(W \times M)$, and that the normal cone of $Z \times B \to R^p(Z \times B)$ is $C \times D$.

By Corollary 6.5, it remains to show that

$$(-1)^{(r+s)+(n+m)} b(-T_{W \times M}|_{X \times Y})((\widetilde{C \times D})_{(E \times V)})$$
$$= (-1)^{r+n} b(-T_W|_X)(\widetilde{C}_E) \times (-1)^{s+m} b(-T_M|_Y)(\widetilde{D}_V)$$

Since b is multiplicative, it is enough to show that

$$(\widetilde{C \times D})_{(E \times V)} = \widetilde{C}_E \times \widetilde{D}_V$$

but this follows from the fact that $[C] \times [D] = [C \times D] \in \mathrm{Ch}^G(E \times V)$. $\square$

7.2. Functorialities.

Lemma 7.4. *Let $q : \mathbb{P}^r \times X \to X$ be the projection. Then $q_* \circ S^{\mathbb{P}^r \times X} = S^X \circ q_*$.*

Proof. The group $\mathrm{CH}(\mathbb{P}^r \times X)$ is generated by the cycles $\gamma = [\mathbb{P}^k \times Z]$ for all closed subvarieties $Z \subset X$ and $k \leq r$. Hence by Proposition 5.7 we may assume $Z = X$ and $k = r$. The statement is obvious if $r = 0$, so we may assume $r > 0$. Since $q_*(\gamma) = 0$ we need to prove that $q_* S^{\mathbb{P}^r \times X}(\gamma) = 0$.

By Theorem 7.3, we have that

$$S^{\mathbb{P}^r \times X}(\gamma) = S^{\mathbb{P}^r}([\mathbb{P}^r]) \times S^X([X])$$

but by Proposition 7.2, $S^{\mathbb{P}^r}([\mathbb{P}^r]) = b(-T_{\mathbb{P}^r})([\mathbb{P}^r]) = (1 + h^{p-1})^{-r-1}$, where h is the class of a hyperplane in $\mathbb{P}^r$. Therefore,

$$q_* S^{\mathbb{P}^r \times X}(\gamma) = \deg(1 + h^{p-1})^{-r-1} \cdot S^X([X])$$

Now $\deg(1 + h^{p-1})^{-r-1} = 0$ unless $r = (p-1)k$ for some integer k, and in this case,

$$\deg(1 + h^{p-1})^{-r-1} = \binom{-r-1}{k} = \binom{-(p-1)k-1}{k} = (-1)^k \binom{pk}{k}$$

which is divisible by p if $k > 0$. $\square$

Theorem 7.5. *Let $f : X \to Y$ be a proper morphism. Then the following diagram commutes:*

$$
\begin{array}{ccc}
\mathrm{Ch}(X) & \xrightarrow{\ S^X\ } & \mathrm{Ch}(X) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{Ch}(Y) & \xrightarrow{\ S^Y\ } & \mathrm{Ch}(Y)
\end{array}
$$

Proof. f factors as the composition of a closed embedding $X \to \mathbb{P}^r \times Y$ and the projection $\mathbb{P}^r \times Y \to Y$, so this result follows from Proposition 5.7 and Lemma 7.4. $\square$

Proposition 7.6. *Let $f : Y \to X$ be a regular closed embedding of schemes of codimension s. Let $V \to Y$ and $W \to X$ be vector bundles with $V^G = Y$ and $W^G = X$. Let $k : \overline{U}_V \to \overline{U}_W$ be the induced closed embedding. Then*

$$\rho^V \circ k^*(\gamma) = \mu(W|_Y / V) \circ f^* \circ \rho^W(\gamma)$$

for all $\gamma \in \mathrm{Ch}(\overline{U}_W)$. In particular, if both X and Y are smooth, and if W is the normal bundle of $X \to R^p(X)$ and V is the normal bundle of $Y \to R^p(Y)$,

$$\rho^V \circ k^*(\gamma) = (-1)^s \omega(N) \circ f^* \circ \rho^W(\gamma)$$

where N is the normal bundle of Y in X.

Proof. Consider the diagram

$$
\begin{array}{ccccc}
\overline{U}_V & \xrightarrow{\ i\ } & \overline{U}_E & \xrightarrow{\ j\ } & \overline{U}_W \\
\downarrow & & \downarrow & & \downarrow \\
Y & = & Y & \xrightarrow{\ f\ } & X
\end{array}
$$

where $E = W|_Y$. Then, by Corollary 4.8 and Lemma 6.1,

$$\rho^V(k^*(\gamma)) = \rho^V(i^*(j^*(\gamma))) = (\mu(E/V) \circ \rho^E)(j^*(\gamma)) = (\mu(E/V) \circ f^* \circ \rho^W)(\gamma)$$

Now consider the case where W is the normal bundle of $X \to R^p(X)$ and V is the normal bundle of $Y \to R^p(Y)$; we proceed as in the proof of Corollary 6.3. As before, we set H to be the cokernel of the embedding $F \to F[G]$; then $W = T_X \otimes H$, $V = T_Y \otimes H$, and

$W|_Y/V = N \otimes H$. Note that s is the rank of N as a vector bundle over Y. Suppose N has a filtration with quotients N_i ($i = 1, 2, \ldots s$) and H has a filtration with quotients H_j ($j = 1, 2, \ldots p - 1$); note that the H_j are trivial line bundles. Hence $N \otimes H$ has a filtration with quotients $N_i \otimes H_j$, where G acts with weight j, and so

$$
\begin{aligned}
\mu(N \otimes H) &= \prod_{j=1}^{p-1}\prod_{i=1}^{s}(j + c_1(N_i \otimes H_j)) \\
&= \prod_{j=1}^{p-1}\prod_{i=1}^{s}(j + c_1(N_i) + c_1(H_j)) \\
&= \prod_{i=1}^{s}\prod_{j=1}^{p-1}(j + c_1(N_i)) \\
&= \prod_{i=1}^{s}(-1 + c_1(N_i)^{p-1}) \\
&= (-1)^s \prod_{i=1}^{s}(1 - c_1(N_i)^{p-1}) \\
&= (-1)^s \omega(N)
\end{aligned}
$$

$\square$

Proposition 7.7. *Let* $f : Y \to X$ *be a closed embedding of smooth schemes with normal bundle* N. *Then*

(7)
$$(b(N) \circ f^* \circ S^X)(\gamma) = (S^Y \circ f^*)(\gamma)$$

for all $\gamma \in \mathrm{Ch}(X)$.

Proof. We may assume $\gamma = [Z]$ for some closed subvariety $Z \subset X$. Let $n = \dim(Z)$, s the codimension of f, and $r = n - s = \dim f^*Z$. Let $k : \overline{U}_Y \to \overline{U}_X$ be the morphism induced by f. Then, by Propositions 5.5 and 7.6:

$$
\begin{aligned}
&\Big(\sum_{i+j=k} b_{i(p-1)}(N) \circ f^* \circ S_j^X\Big)([Z]) \\
&= \Big(\sum_{i+j=k} (-1)^{n+j} b_{i(p-1)}(N) \circ f^* \circ \rho_{(p-1)(n+j)}^X \circ P_X\Big)([Z]) \\
&= \Big(\sum_{i+j=k} (-1)^{n+j}(-1)^i \omega_{i(p-1)}(N) \circ f^* \circ \rho_{(p-1)(n+j)}^X \circ P_X\Big)([Z]) \\
&= \Big(\sum_{i+j=k} (-1)^{n+j}(-1)^i(-1)^s \rho_{(p-1)(r+k)}^Y \circ k^* \circ P_X\Big)([Z]) \\
&= \Big(\sum_{i+j=k} (-1)^{n-s}(-1)^{i+j} \rho_{(p-1)(r+k)}^Y \circ P_Y \circ f^*\Big)([Z]) \\
&= (-1)^{r+k}(\rho_{(p-1)(r+k)}^Y \circ P_Y \circ f^*)([Z]) \\
&= (S_k^Y \circ f^*)([Z])
\end{aligned}
$$

$\square$

8. Steenrod operations on smooth schemes

Definition. Let X be a smooth scheme. We define the *Steenrod operations of cohomological type* by the formula

$$S_X = b(T_X) \circ S^X$$

We write $S_X = \coprod S_X^k$, where for each k, S_X^k is an operation

$$S_X^k : \mathrm{Ch}^n(X) \to \mathrm{Ch}^{n+k(p-1)}(X)$$

Proposition 8.1. *(Wu Formula) Let X be a smooth scheme, $Z \subset X$ a smooth closed sub-scheme. Then $S_X([Z]) = i_* b(N)([Z])$, where N is the normal bundle of the closed embedding $i : Z \to X$.*

Proof. By Proposition 7.2, we have

$$
\begin{aligned}
S_X([Z]) &= (b(T_X) \circ S^X)([Z]) \\
&= (b(T_X) \circ i_* \circ b(-T_Z))([Z]) \\
&= (i_* b(i^* T_X) \circ b(-T_Z))([Z]) \\
&= i_* b(N)([Z])
\end{aligned}
$$

$\square$

Theorem 8.2. *Let X and Y be two smooth schemes. Then $S_{X \times Y} = S_X \times S_Y$.*

Proof. Since $T_{X \times Y} = T_X \times T_Y$, we have by Theorem 7.3 that

$$S_{X \times Y} = b(T_{X \times Y}) \circ S^{X \times Y} = b(T_X) S^X \times b(T_Y) S^Y = S_X \times S_Y$$

$\square$

Theorem 8.3. *Let $f : Y \to X$ be a morphism of smooth schemes. Then $f^* \circ S_X = S_Y \circ f^*$.*

Proof. Suppose first that f is a closed embedding with normal bundle N. Then by Proposition 7.7,

$$
\begin{aligned}
f^* \circ S_X &= f^* \circ b(T_X) \circ S^X \\
&= b(f^* T_X) \circ f^* \circ S^X \\
&= b(T_Y) \circ b(N) \circ f^* \circ S^X \\
&= b(T_Y) \circ S^Y \circ f^* \\
&= S_Y \circ f^*
\end{aligned}
$$

Second, suppose f is a projection $f : Y \times X \to X$. Let $Z \subset X$ be a closed subvariety. We have that $f^*([Z]) = [Y \times Z] = [Y] \times [Z]$, so, using Proposition 7.2 and Theorem 8.2:

$$
\begin{aligned}
S_{Y \times X}(f^*([Z])) &= S_{Y \times X}([Y \times Z]) = S_{Y \times X}([Y] \times [Z]) \\
&= S_Y([Y]) \times S_X([Z]) \\
&= (b(T_Y) S^Y([Y])) \times S_X([Z]) \\
&= [Y] \times S_X([Z]) \\
&= f^* S_X([Z])
\end{aligned}
$$

In general, we can write f as the composition of the closed embedding $Y \to Y \times X$ and the projection $Y \times X \to X$, so the result follows from the two above cases. $\square$

Corollary 8.4. *(Cartan Formula) Let X be a smooth scheme. Then*

$$S_X(\gamma \cdot \delta) = S_X(\gamma) \cdot S_X(\delta)$$

for any $\gamma, \delta \in \mathrm{Ch}(X)$.

Proof. Let $i : X \to X \times X$ be the diagonal embedding. Then by Theorems 8.2 and 8.3,

$$
\begin{aligned}
S_X(\gamma \cdot \delta) = S_X(i^*(\gamma \times \delta)) &= i^*(S_{X \times Y}(\gamma \times \delta) \\
&= i^*(S_X(\gamma) \times S_X(\delta)) \\
&= S_X(\gamma) \cdot S_X(\delta)
\end{aligned}
$$

$\square$

Theorem 8.5. *Let X be a smooth scheme. Then for any $\delta \in \mathrm{Ch}^k(X)$,*

$$S_X^r(\delta) = \begin{cases} \delta, & r = 0 \\ \delta^p, & r = k \\ 0, & r < 0 \text{ or } r > k \end{cases}$$

Proof. By definition and Theorem 7.1, $S_X^r = 0$ if $r < 0$ and S_X^0 is the identity function. We assume that X is a variety and that $\delta = [Z]$ for a closed subvariety $Z \subset X$. Let $d = \dim X, n = \dim Z$ so that $k = d - n$. In the notation of Corollary 6.5, we have

$$S_X(\delta) = b(T_X)b(-T_X)(\widetilde{C}_E) = \widetilde{C}_E = \sum \tilde{\gamma}_i$$

Since $\tilde{\gamma}_i \in \mathrm{Ch}_{n-i(p-1)}(X), S_X^r(\delta) = \tilde{\gamma}_r$. Following the notation of Lemma 6.4, $\tilde{\gamma}_r = (-1)^{d+n+r} a_{(d-n-r)(p-1)}$, and since $a_j = 0$ for $j < 0$, $S_X^r(\delta) = 0$ if $r > d - n = k$.

It remains to show that $S_X^k(\delta) = \delta^p$. Since $S_X^k(\delta) = \tilde{\gamma}_k = a_0$, we need to show that $\delta^p = a_0$. By a restriction-corestriction argument, we may assume that F contains p-th roots of unity, so that $R^p(X) = X^p$. Let $\Delta : X \to X^p$ be the diagonal embedding, C the normal cone of $Z \to Z^p$, and N the normal bundle of $X \to X^p$. Let $h : C \to N$ be the closed embedding, and $s : N \to X$ the projection. By [4], Proposition 52.6,

$$s^*(\delta^p) = s^*\Delta^*([Z^p]) = \sigma_\Delta([Z^p]) = h_*([C])$$

where σ_Δ is the deformation homomorphism. Now consider the diagram

(8)
$$\begin{array}{ccc} \mathrm{Ch}^G(N) & \longrightarrow & \mathrm{Ch}(N) \\ \uparrow{s_G^*} & & \uparrow{s^*} \\ \mathrm{Ch}^G(X) & \longrightarrow & \mathrm{Ch}(X) \end{array}$$

where the horizontal maps "forget the G-action" on N and X (to be precise, these maps are the transfer from G to the trivial subgroup; see [1], 3.4 for a detailed explanation). We have that (8) commutes, as the horizontal maps are induced by a push-forward. Consider $C_N^G \in \mathrm{Ch}^G(X)$; if we follow it right and up, we get $s^*(a_0)$; if we follow it up and then right, we get $h_*([C])$. Hence $s^*(a_0) = h_*([C]) = s^*(\delta^p)$, and therefore $\delta^p = a_0$, as s^* is an isomorphism. $\square$

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